Do Large-Firm Bargaining Models Amplify and Propagate Aggregate Productivity Shocks?

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This version: March 2, 2011
First version: October 2010

Abstract

Benchmark labor search models abstract from the large cross-sectional heterogeneity in firm size and employment growth distributions present in US data; these models have also struggled to generate empirically-plausible amplification and propagation of productivity shocks. Does accounting for firm heterogeneity help in solving this problem? Recent work by several authors has argued that the slow adjustment of the firm size distribution following a shock to the economy might help generate more persistent or more volatile responses of key endogenous variables such as vacancies or unemployment than in the Mortensen-Pissarides benchmark. I study a model that allows for very rich microeconomic heterogeneity of firm productivity and productivity growth. I show that despite the ability of the model to replicate key cross-sectional features of the employment and employment growth distribution across firms, the equilibrium behavior of aggregate variables such as unemployment and the vacancy-to-unemployment ratio is practically indistinguishable from that in an appropriately-calibrated Mortensen-Pissarides model. In particular, unemployment is only volatile if the surplus from employment is small, and the vacancy-to-unemployment ratio is a jump variable. Despite allowing for both transitory and permanent idiosyncratic heterogeneity in firm productivity as well as for aggregate productivity shocks, the model is tractable enough to be solved without the need for approximate aggregation solution methods.

JEL Codes: E24, J41, J64.

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1 Introduction

Two major empirical shortcomings of the benchmark Mortensen-Pissarides search-and-matching model of the labor market (hereafter, the MP model) in accounting for the cyclical behavior of key aggregate variables are now familiar. First, under a parameterization in which the surplus an employed worker enjoys relative to an unemployed worker is plausibly large, model-generated time series for unemployment and for the vacancy-unemployment ratio are not nearly as volatile as observed in US data (Shimer, 2005, 2010; Hagedorn and Manovskii, 2008). Second, while in US data the response of the vacancy-unemployment ratio to productivity shocks is hump-shaped, in the MP model no suitable propagation mechanism exists: the vacancy-unemployment ratio is a jump variable (Fujita and Ramey, 2007). Another feature of the benchmark MP framework is that it abstracts from the notion of firm size, so that it is not possible to use the framework to account for cross-sectional patterns of firm-level employment and firm-level employment growth. Recently, several authors (Acemoglu and Hawkins, 2010; Elsby and Michaels, 2010; Kaas and Kircher, 2011) have studied models of frictional labor markets in which firms employ multiple workers and operate production technologies with decreasing returns to labor, and found that the behavior of aggregate variables are different in these models from the benchmark MP model. Does the presence of a nondegenerate distribution of employment across firms, which will respond only sluggishly to aggregate shocks, provide the propagation mechanism missing from the basic model? Does it by itself enhance the ability of the model to account for large fluctuations in unemployment in the presence of small shocks to labor productivity?

In this paper, I answer these questions in the negative. I study a model of a frictional labor market that is consistent with rich cross-sectional heterogeneity in employment and employment growth across establishments, and show that it can be calibrated to match the cross-sectional distributions of these variables in US data. However, when it comes to accounting for the response of unemployment and of the vacancy-unemployment ratio to shocks to labor productivity, the model exhibits the familiar properties of the MP model. If recruiting is a time-intensive activity and if preferences take a balanced-growth form so that income and substitution effects of productivity shocks on labor supply cancel, then, analogously to the result of Shimer (2010) for the MP model, unemployment and the vacancy-unemployment ratio do not vary with labor productivity. If I instead make the assumption that is more customary in the labor search literature in which recruiting is goods-intensive and households have linear preferences over consumption, then analogously to the results of Shimer (2005) and Hagedorn and Manovskii (2008), the unemployment rate and the vacancy-unemployment ratio do vary with aggregate productivity; however, the model does not have additional explanatory power relative to the MP benchmark. Unemployment is not very volatile unless the employment surplus is small, and the vacancy-unemployment ratio is a jump variable that directly inherits the properties of the stochastic process for labor productivity without any additional propagation. In summary, the omission of productive heterogeneity of firms in the MP model does not by itself account for the failure of that model to amplify or propagate productivity shocks. Even though in my model the firm-size distribution is a state variable and
responds only slowly to aggregate shocks, it generates neither propagation nor amplification.

An important contribution of the paper is that despite the rich microeconomic heterogeneity – I allow for firms to have heterogeneous productivity with both transitory and permanent components, as well as for an aggregate productivity shock – the model is nearly solvable in closed form. Conditional on a guess for the relationship between aggregate labor productivity and the equilibrium vacancy-unemployment ratio, solving the model essentially reduces to solving a system of linear equations. In particular, there is no need for approximate aggregation techniques (Krusell and Smith, 1998) and or for the requisite assumption that economic agents are boundedly rational in order to compute the equilibrium. The only caveat for this result is that the equilibrium I compute does not apply if it is possible for shocks to aggregate productivity to be sufficiently negative that firm entry ceases completely for some time, a possibility that seems irrelevant if the model is calibrated to match the empirical behavior of establishment entry in the US economy.

I now explain some of the key features of the model. The technology operated by firms exhibits decreasing returns to labor; in a fuller model these decreasing returns might arise from the presence of fixed factors such as physical capital, from imperfect substitutability for consumers of the goods produced by different firms, or from managerial span-of-control as in Lucas (1978). Firms are heterogeneous in productivity, and this heterogeneity can include both permanent and transitory components. The stochastic nature of firm-level productivity allows the model to account for the empirical cross-sectional patterns of firm employment levels and growth rates.

Matching between firms and workers in the labor market is frictional, and the search process is random. Firms post vacancies in order to hire workers, and the vacancy-posting technology exhibits constant returns at the firm level both in terms of cost and of the number of workers contacted. However, as in the MP model, when the aggregate vacancy-unemployment ratio is high, the yield of contacted workers per vacancy declines, so that a firm needs to post more vacancies in order to hire the same number of workers. The fact that hiring workers is costly (although firing them is not) means that firms do not always alter their employment levels in response to productivity shocks, and allows for the firm-size distribution to respond only sluggishly to aggregate productivity shocks. (However, as already explained, a key contribution of the paper is to show that despite the persistent response of the firm size distribution to productivity shocks, this does not propagate to sluggish adjustment of market tightness.)

Wages are determined according to the generalization of Nash bargaining to the case of many-to-one matching proposed in the labor context by Stole and Zwiebel (1996a,b) and Smith (1999). In the Stole-Zwiebel framework, a firm bargains with each worker as if she were marginal, and in this bargaining process the outside option of the worker is unemployment and the outside option of the firm is to recommence the bargaining process with one fewer worker. This framework is by now the benchmark framework for wage determination in the context of random-search models of the labor market, and has been applied to study contracting and technological adoption (Acemoglu et al., 2007), wage determination (Roys, 2010), the interaction of product market regulation and the labor market (Felbermayr and Prat, 2007; Delacroix and Samaniego, 2009; Ebell and Haefke, 2007).
and the labor market effects of changes in trade patterns (Coşar et al., 2010; Helpman and Itskhoki, 2010). Although it is well-known that the Stole-Zwiebel bargaining process does not generate a constrained efficient outcome (firms hire inefficiently many workers so as to drive down wages of inframarginal colleagues), this inefficiency does not drive the results presented in this paper. More precisely, I show in Hawkins (2011) that the dynamics of both aggregate variables like unemployment and the vacancy-unemployment ratio and of idiosyncratic variables like firm-level employment in the model presented in the current paper are precisely identical to those in the constrained efficient allocation of an economy that differs only in the average level of productivity.

I defer a full discussion of how the paper is related to the literature to Section 5; however, I briefly explain here how the model differs from three recent contributions mentioned above, each of which finds greater persistence in the response to productivity shocks in a multi-worker frictional labor market model than in the MP benchmark. First, the current paper assumes a constant returns to scale vacancy-posting technology, while in both Acemoglu and Hawkins (2010) and Kaas and Kircher (2011) a firm that wishes to hire twice as many new workers must pay more than twice the cost. It is natural to conjecture that it is these convex costs of hiring that generate the sluggish aggregate responses to productivity shocks in those two papers, particularly when such a convex hiring cost function already generates significantly more persistence in the benchmark MP framework (Yashiv, 2006). The key difference between the present paper and Elsby and Michaels (2010), on the other hand, is that I allow for vacancy creation to take place both on the intensive margin (additional hiring by incumbent firms) and on the extensive margin (entry of new firms). Elsby and Michaels allow only for the intensive margin. Thus, in their model, new hiring in response to a positive productivity shock must entirely be made by incumbent firms, but because of the decreasing returns to labor, this generates a qualitatively different response of aggregate variables to productivity shocks to that in the present model. (The model studied in this paper is closely related to that of Elsby and Michaels in other respects.)

The structure of the remainder of the paper is as follows. In Section 2 I study the version of the model with balanced growth preferences and a time-intensive vacancy-posting technology; in this version productivity shocks do not induce fluctuations in unemployment. Section 3 modifies the benchmark model to introduce linear preferences over consumption and goods-intensive vacancy-posting. In Section 4 I undertake a quantitative analysis of the model in Section 3, and relate its findings to those of the MP model. Section 5 relates the model and the results of the paper to the literature, and discusses possible extensions. Section 6 concludes.

2 Model with Balanced Growth Preferences

In this section I introduce the benchmark model and prove that an equilibrium in which consumption and wages are proportional to current productivity and unemployment is constant exists. I then provide an algorithm for computing this equilibrium.

Time is discrete, $t = 0, 1, 2, \ldots$. There are four types of agents in the economy, a representative
household containing a continuum of measure 1 of workers and continua of three types of firms, production firms, recruitment firms, and construction firms. Only some production firms are active at any time. For the sake of brevity, the term ‘firm’ in the sequel refers to production firms unless otherwise specified. Each period, the following events occur.

- Nature draws aggregate productivity and, for each active production firm, idiosyncratic productivity. These values are publicly observable.
- Incumbent production firms choose how many of their existing workers to fire.
- Entry of new production firms occurs and their idiosyncratic productivity is realized.
- Incumbent and new entrant production firms engage recruitment firms in order to hire unemployed workers; meetings between recruitment firms and unemployed workers occur.
- Production occurs and wages are paid.
- Some firms are exogenously destroyed.

I now discuss the economic agents and each of the listed events in more detail.

### 2.1 Productivity and Production Firms

There is a large continuum of production firms. Production firms are ultimately owned by the representative household, via a mutual fund which holds a diversified portfolio of all production firms. Any profits earned by the production firms are rebated as dividends to the representative household.

At the beginning of each period, active production firms learn the new realizations of the aggregate productivity shock and of their own idiosyncratic productivity shock. I therefore commence by describing the assumptions on the productivity shock process. Productivity shocks affect production firms only. I assume that when the aggregate productivity shock is \( p \) and its own idiosyncratic productivity shock is \( \pi \), a production firm with \( n \) employees produces output per period of \( y(n; p, \pi) = p\pi y(n) \), where \( y(n) \) is strictly increasing, strictly concave, and satisfies the usual Inada conditions.

To simplify notation and to enable me to provide a constructive algorithm to calculate equilibrium, it is convenient to assume that the aggregate productivity shock \( p \) takes only finitely many possible values, \( z_1 < z_2 < \ldots < z_m \). Denote by \( p^t = (p_0, p_1, \ldots, p_t) \) the history of aggregate productivity shocks, and by \( P^p(p^t) \) the probability, before the realization of the period-0 productivity shocks, of history \( p^t \) being realized through period \( t \). I assume complete markets for claims on the consumption good contingent on aggregate variables, so it is helpful to denote by \( s^t = (s_0, s_1, \ldots, s_t) \) the history of the economy, and by \( P(s^t) \) the period-0 probability of history \( s^t \) being realized through period \( t \). In addition to the aggregate productivity shock, \( s_t \) incorporates all payoff-relevant variables such as the distribution of employment across firms, prices and wages, and
the consumption, asset holdings and labor supply of households. In fact I will study an equilibrium in which the history of aggregate productivity (along with period-0 initial conditions) is a sufficient statistic for the state of the economy, but I do not wish to assume this structure from the outset and therefore keep track of \( s^t \) and not merely \( p^t \).

Denote by \( \pi^t = (\pi_{0t}, \pi_{1t}, \ldots, \pi_{lt}) \) the history of idiosyncratic productivity shocks for an arbitrary production firm (indexed by \( l \)). It simplifies the analysis to assume that the idiosyncratic productivity shock process is iid across firms (conditional on the history of idiosyncratic productivity shocks), orthogonal to the aggregate productivity shock, and the same for entrants and incumbents, although nothing material is altered by relaxing any of these assumptions except for the notational burden.\(^1\) If a firm becomes active during period \( \tau \geq 0 \) after history \( s^\tau \), I write \( \pi_{0t} = 0 \) for \( t < \tau \), and \( \Pi(\pi^\tau; s^\tau) \) for the probability, before the realization of the period-\( \tau \) value of the idiosyncratic productivity shock, of the history \( \pi^t \) through date \( t \). Note that by assumption \( \Pi(\pi^\tau; s^\tau) = 0 \) if \( \pi_\sigma > 0 \) for any \( \sigma < \tau \). The distribution of idiosyncratic productivity shocks that can be drawn by new entrants (that is, the set of probabilities \( \Pi((0, \ldots, 0, \zeta); s^\tau) \) for \( j = 1, 2, \ldots, m_\sigma \) could potentially be different than the marginal distribution of the new idiosyncratic shock in period \( t \) for earlier entrants or for firms already active at period 0. I denote this distribution by \( F^\tau_\pi(\pi) \), and assume that it is independent of the period \( \tau \) and history \( s^\tau \).\(^2\)

It is convenient to assume that the idiosyncratic shock takes only finitely many values, \( \zeta_1 < \zeta_2 < \ldots < \zeta_{m_\sigma} \) and follows a first-order Markov process which is identical across firms (including for initially inactive firms once they become active); this assumption simplifies notation by allowing for a unified treatment of all firms, regardless of for how long a particular firm has been active.

Write \( \Pi(\pi^t) \) for the probability, before the realization of the period-0 idiosyncratic productivity shocks, of the history \( \pi^t \) through date \( t \) for any initially-active firm, independently across firms. The first-order Markov assumption implies that it is possible to write

\[
\frac{\Pi((\pi^{t-1}, \pi_t, \pi_{t+1}))}{\Pi((\pi^{t-1}, \pi_t))} = \frac{\Pi((\pi^{t-1}, \pi_t, \pi_{t+1}), s^\tau)}{\Pi((\pi^{t-1}, \pi_t), s^\tau)} = \mu(\pi_t, \pi_{t+1})
\]

for some function \( \mu(\cdot, \cdot) \).

Note that I allow for arbitrary history dependence in the aggregate productivity process, al-

\(^1\)If entrants have a different productivity process to incumbents after the period of entry, then this may act like an aggregate productivity shock; the difficulty is that the notational burden is increased since I have to write separate notation for initially-active firms and for entrants even after the period of entry. However, nothing material would be altered by relaxing this assumption.

\(^2\)One possible natural assumption on \( F^\tau_\pi(\cdot) \), following Mortensen and Pissarides (1994), is that potential entrants know their productivity before entering, in which case only those firms with productivity \( \pi_{m_\sigma} \) will enter in equilibrium. An alternative is to assume that potential entrants draw from the ergodic distribution of idiosyncratic shocks for active firms. A final possible assumption is that \( F^\tau_\pi \) is concentrated on low values of \( \pi \), which means that entrants would increase in expected productivity over time. This would be consistent with evidence from Haltiwanger et al that new entrant firms are initially smaller than incumbents; cf Lee and Mukoyama also. *** find citations
though in the calibrated model I will assume that the aggregate productivity shock processes, like that for idiosyncratic shocks, is first-order Markov.

Finally, I assume that all the realizations of the aggregate and idiosyncratic shock process are sufficiently high that no firm ever wishes to exit endogenously.\footnote{Note that I allow for exogenous firm destruction with probability $\delta$ per period iid across firms and histories. The process of exogenous firm destruction can be reinterpreted as arising when a firm draws a low enough idiosyncratic productivity shock (specifically, zero) which it expects to be an absorbing state. Thus, the chief simplification afforded by the assumption that there is no endogenous exit is notational, since the exit probability is independent of aggregate and idiosyncratic productivity. It is possible to relax this assumption at the cost of significantly increasing the notational burden; the main results of the paper carry over unchanged to such an environment.}

The first action available to firms each period is that they may lay workers off in the event of a sufficiently negative shock. I assume that firms are able to lay workers off at will and at zero cost.\footnote{Positive firing costs paid by firms make the bargaining assumption to be imposed below inappropriate, since workers who would be fired if not for a positive firing cost are paid less than their outside option and would therefore voluntarily quit. If workers were able to commit not to quit, the extension to the case of positive firing costs is not difficult.}

Next, each period, after the aggregate shock has been revealed and firing has occurred, all inactive firms have the option of becoming active by buying a factory. Factories are produced by construction firms (to be described below) for a price $k^f(s^t)$ units of the history-$s^t$ consumption good. Inactive production firms simply wait for the next period. Having paid the entry cost, a newly-active production firm observes its idiosyncratic productivity shock.

After factories have been purchased, incumbent and new entrant production firms alike pay recruitment firms in order to hire unemployed workers. The opportunity to match with an unemployed worker can be purchased from recruitment firms (to be described below) for a price $k^r(s^t)$ units of the history-$s^t$ consumption good. Once a production firm has paid this cost, bargaining between the firm and its employees, incumbent and newly-recruited, begins (to be described further below). Denote by $w(s^t, a(s^t); \pi^t, n)$ the wage (in units of the history-$s^t$ consumption good) negotiated between the firm and its employees when it enters production with $n$ workers, the history of the economy is $s^t$, the representative household has assets $a(s^t)$, and the history of idiosyncratic productivity is $\pi^t$.

After production has occurred and wage payments have been made, the last event in each period is that a fraction $\delta$ of active production firms are destroyed at the end of each period. Such a firm is removed from the economy and replaced by a new inactive firm. Its workers enter the beginning of the following period unemployed.

That is, the objective of a firm that begins period 0 active and with $n-1$ employees, with the history of the economy given by $s^0$ and its own idiosyncratic shock $\pi_0$ is to maximize

$$J(n-1; s^0, \pi^0) = \sum_{t=0}^{\infty} \sum_{s^t} \sum_{\pi^t} q_0(s^t)(1 - \delta)^t \Pi(\pi^t) \left[ - h_t(n-1, s^t, \pi^t)k^f(s^t) + p_t\pi_t y(n(n-1, s^t, \pi^t)) \right. \\
\left. - n(n-1, s^t, \pi^t)w(s^t, a(s^t); \pi^t, n(n-1, s^t, \pi^t)) \right].$$

Here $q_0(s^t)$ denotes the Arrow-Debreu price of goods in period $t$ after history $s^t$ (the numeraire is
the date-0, history $s^0$ good). The law of motion for the firm’s employment can be written

$$n(n_{-1}, s^t, \pi^t) = n(n_{-1}, s^{t-1}, \pi^{t-1}) - f(n_{-1}, s^t, \pi^t) + h_t(n_{-1}, s^t, \pi^t),$$

(3)

where $f(n_{-1}, s^t, \pi^t) \geq 0$ denotes the firm’s choice of how many workers to fire in period $t$ after history $(s^t, \pi^t)$ and $h_t(n_{-1}, s^t, \pi^t) \geq 0$ denotes the corresponding hiring choice. (Since no information is revealed between the decision on firing and that on recruitment, it is immediate that at most one of these two variables will be non-zero at any time for any production firm.)

The objective of a firm that begins period 0 inactive (and therefore with no employees) is similar to (2), but notationally more complex since such a firm additionally has to decide after what aggregate histories $s^\tau$ to become active. More precisely, an initially inactive firm maximizes

$$J^f(s^0) = \sum_{\tau=0}^{\infty} \sum_{s^\tau} \left[-q_0(s^\tau)k^f(s^\tau) + J^e(s^\tau)\right],$$

where $J^e(s^\tau)$, the value of a new entrant firm in period $\tau$ after history $s^\tau$ after having paid for its factory but before learning its idiosyncratic productivity shock, is given by

$$J^e(s^\tau) = \sum_{t=\tau}^{\infty} \sum_{s^t} \sum_{\pi^t} q_0(s^t)(1-\delta)^{t-\tau} \Pi_\pi(\pi^t) \left[-h_t(s^\tau, s^t, \pi^t)k^e(s^\tau) + p_{t} \Pi_{\pi} y(n_t(s^\tau, s^t, \pi^t)) - n_t(s^\tau, s^t, \pi^t)w(n_t(s^\tau, s^t, \pi^t))\right].$$

The summation over $s^t$ is taken only over those $s^t$ of which $s^\tau$ is an initial segment, and that over $\pi^t$ only over sequences with first $t-1$ elements equal to zero. The law of motion for the number of employees satisfies a constraint which takes the same form as (3), that is,

$$n_t(s^\tau, s^t, \pi^t) = n_t(s^\tau, s^{t-1}, \pi^{t-1}) - f_t(s^\tau, s^t, \pi^t) + h_t(s^\tau, s^t, \pi^t),$$

and subject to the constraint that hiring and firing must be nonnegative (that is, $f_t(s^\tau, s^t, \pi^t), h_t(s^\tau, s^t, \pi^t) \geq 0$).

It is convenient to write the firm’s problem recursively. To do so, denote by $\hat{J}(n_0; s^t, \pi_t)$ the value of a firm which arrives with $n_0$ workers at the beginning of period $t$, before firing and recruitment occur but after the aggregate and idiosyncratic productivity shocks are revealed, and by $J(n; s^t, \pi^t)$ the value of a firm which has $n$ workers after the recruitment season is over in period $t$. These values are to be measured in terms of date $t$, history $s^t$ goods. Then

$$\hat{J}(n; s^t, \pi^t) = \max_{f, h \geq 0} \left[-k^e(s^t)h + J(n - f + h; s^t, \pi^t)\right],$$

(4)

and

$$J(n; s^t, \pi^t) = p_t \Pi_{\pi} y(n) - nw(n; s^t, \pi^t) + (1-\delta) \sum_{s^{t+1}} q_t(s^{t+1}) \sum_{\pi^{t+1}} \mu(\pi_t, \pi_{t+1})\hat{J}(n; s^{t+1}, \pi^{t+1})$$

(5)
Here $q_t(s^{t+1})$ denotes the period-$t$ price of a unit of consumption goods at date $t + 1$. Equations (4) and (5) give a recursive representation of the problem for a production firm. It is convenient to write $f(n; s^t, \pi^t)$ and $h(n; s^t, \pi^t)$ for the optimal firing and recruitment policies of the firm, and

\[ n'(n; s^t, \pi^t) = n - f(n; s^t, \pi^t) + h(n; s^t, \pi^t) \]  

(6)

for the law of motion for the firm’s employment.

Note that because of (1), the recursive representation of the firm’s problem applies independently of whether a firm was already active in period 0 or entered after some history $s^\tau$ that preceded $s^t$. I will therefore in future not write notation for the behavior of entrant firms separate from that for initially-active firms.

There is free entry for production firms, which requires that

\[ \sum_j F^n(\pi_j) \hat{J}(0; s^t, j) \leq k_f(p^t) \]  

(7)

with equality if a positive amount of entry occurs.

Last, since it will be needed below, introduce the notation $G(n; s^t, \pi_t)$ for the cumulative distribution (conditional on history $s^t$ and current idiosyncratic productivity shock $\pi_t$) of the number of firms with employment less than or equal to $n$ during the production phase of period $t$.

### 2.2 Construction Firms

An inactive production firm needs to purchase a factory from a construction firm in order to enter. The construction firm sector consists of many firms each of which operates a linear technology which uses $\kappa$ units of labor services to produce a single factory. These firms are not subject to search frictions, and rent their labor each period in a competitive market. Because the construction of factories is finished before the labor market operates, any worker in the economy is able to work in the construction sector during any period in addition to any other activities that she may be engaged in (such as work in the production sector). Denote by $w^f(s^t)$ the wage paid by a construction firm; then the assumption of perfect competition ensures that the price of a factory satisfies

\[ k^f(s^t) = \kappa w^f(s^t). \]  

(8)

### 2.3 Recruitment Firms

Production firms need to purchase the opportunity to meet with new workers from recruitment firms. The recruitment firm sector consists of many firms, each of which operates a technology that produces $\frac{1}{\gamma}vm(\theta)$ recruitment opportunities if it employs $v$ recruiters and if the aggregate ratio of recruiters to unemployed workers looking for jobs is $\theta$. The probability that a particular unemployed worker is matched to a particular recruiting firm is proportional to the number of recruiters employed by that recruiting firm. As in Shimer (2010), this matching technology represents
a matching function that exhibits constant returns to scale in recruiters and unemployed workers jointly. I assume that $m$ is continuous and nonincreasing, with $\lim_{\theta \to 0} m(\theta) = +\infty$. Denote by $f(\theta) = \theta m(\theta)$ the probability that an unemployed worker successfully meets a recruitment firm. I implicitly assume that a law of large numbers apply, so that although matching is stochastic from the perspective of an unemployed worker, it is deterministic from the point of view of recruitment firms. I assume that $\lim_{\theta \to 0} f(\theta) = 0$ and that $f(\cdot)$ is continuous and strictly increasing.

As in the construction sector, the workers employed as recruiters by recruitment firms are rented each period in a competitive labor market which is not affected by search frictions. Denote by $w^r(s^t)$ the wage paid by a recruitment firm. I assume that the recruitment sector is perfectly competitive, so that the price of a recruitment opportunity is

$$k^r(s^t) = \gamma w^r(s^t)/m(\theta(s^t)).$$

(9)

Because the recruitment sector operates after the construction sector is closed and before the production sector operates, any worker in the economy can work in the recruitment sector independently of other activities she may undertake.

Note that the unemployment pool able to be hired in the current period includes those exogenously separated at the end of the previous period, as well as those fired from their firms at the beginning of the current period. Thus, some unemployment spells have a length of zero. When an existing firm is destroyed (which happens with probability $\delta > 0$ per period, iid across firms and over time), all the workers begin the following period unemployed, and the firm is removed from the economy with no scrap value.

2.4 Households

There is a representative household which contains a continuum of measure 1 of members. Each individual member of the household maximizes expected utility and lives for ever. Each member of the household has time-separable preferences over histories of consumption and labor supply, with the period felicity function given by

$$\log c - e^f \phi^f - e^r \phi^r - e^y \phi^y.$$  

(10)

Here $e^f$ and $e^r$ are nonnegative real numbers which measure the worker’s labor supply respectively in the construction and recruitment sectors. $e^y$, on the other hand, takes the value 1 if the worker works in the production sector and 0 otherwise. (That is, there is an intensive margin of labor supply for the individual worker in the construction and recruitment sectors, but only an extensive margin in the production sector. Note that because these activities are not mutually exclusive, there are no additional restrictions on the values of $e^f$, $e^r$, and $e^y$.) The discount factor for future felicity is $\beta$. The household allocates consumption across its members as it desires and instructs its members on whether to work in the recruitment sector or the construction sector. However, the household is not able to influence the number of its workers that are employed in the production sector.
sector, which is determined from last period’s employment in the production sector according to
the choices of production firms for firing, entry, and recruitment. It is standard that this implies
that the household acts as if it maximizes the utility function
\[ \sum_{t=0}^{\infty} \beta^t P(s^t) \left[ \log c(s^t) - \phi^f e^f(s^t) - \phi^r e^r(s^t) - \phi^y e^y(s^t) \right], \]
where \( e^f(s^t), e^r(s^t), \) and \( e^y(s^t) \) are the average labor supply of household members respectively
to the construction, recruitment, and production sectors during period \( t \) after history \( s^t \). \( e^y(s^t) \)
can be interpreted as the fraction of household members working in the production sector. Note
that, following Merz (1995), there is complete insurance within the household: because of the
separability in (10), the household allocates consumption among its members equally, so as to
equalize the marginal utility of consumption across them.

The household faces the budget constraint that the expected present discounted value of con-
sumption must equal initial assets plus the expected present discounted value of labor income plus
expected dividends:
\[ a_0 = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t q_0(s^t) \left( c(s^t) - e^f(s^t)w^f(s^t) - e^r(s^t)w^r(s^t) - e^y(s^t)Ew(s^t) - d(s^t) \right). \]
Here \( Ew(s^t) \) denotes the average wage earned by each member of the household in the production
sector after history \( s^t \). The household takes the evolution of \( Ew(s^t) \) as given, just as it takes the
evolution of \( e^y(s^t) \) as given.\(^5\)

The household owns, via a mutual fund, shares in the market portfolio of firms; \( d(s^t) \) denotes
the dividends received from this portfolio.

The household’s problem can be written recursively. Denote by \( V(s^t, a) \) the household’s utility
after history \( s^t \) if its assets are \( a \). During period \( t \) the household chooses employment in the
construction and recruitment sectors and purchases of state-contingent consumption for period
\( t+1 \), so that its value can be represented by
\[ V(s^t, a) = \max_{a(s^{t+1}), e^f, e^r} \left[ \log c - \phi^f e^f - \phi^r e^r - \phi^y e^y(s^t) + \beta \sum_{s^{t+1} | s^t} \frac{P(s^{t+1})}{P(s^t)} V(s^{t+1}, a(s^{t+1}) \right] \] (11)
\(^5\)To define \( Ew(s^t) \) more precisely, I need to assume that wages depend only on a firm’s employment \( n \) and current
idiosyncratic productivity shock \( \tau_t \) and on the history \( s^t \). This will be established in Lemma 1 below. In this case
the total wages earned by the household’s members employed in the production sector satisfy
\[ e^y(s^t)Ew(s^t) = \sum_{s^t} \sum_{n} \int_0^{\infty} nw(n; z_i, \zeta_j) dG(n; s^t, \zeta_j). \]
subject to the constraints that \( e^c, e^r \in [0, 1] \), to the intertemporal budget constraint

\[
c = a + e^f w^f(s^t) + e^r w^r(s^t) + E w(s^t) - \sum_{s^{t+1}|s^t} q_t(s^{t+1})a(s^{t+1}),
\]

and taking the evolution of \( e^y(\cdot) \) and \( E w(\cdot) \) as given.

Because of the linear disutility, the household’s optimal choice of labor supply to the construction sector is 0 if \( \phi^f > w^f(s^t)/c(s^t, a) \), where \( c(s^t, a) \) is the consumption choice of a household which enters period \( t \) after history \( s^t \) with assets \( a \). Likewise, it is unbounded if \( \phi^f < w^f(s^t)/c(s^t, a) \), and indeterminate if \( \phi^f = w^c(s^t)/c(s^t, a) \). Assuming that parameters are such that the equilibrium always requires an interior choice of labor supply to the construction sector, then this implies that

\[
\phi^f = \frac{w^f(s^t)}{c(s^t)};
\]

where \( c(s^t) = c(s^t, a(s^t)) \) is the equilibrium value of the representative household’s consumption after history \( s^t \). Analogously, assuming that the equilibrium always requires an interior choice of labor supply to the recruitment sector, it must be that

\[
\phi^r = \frac{w^r(s^t)}{c(s^t)}.
\]

As is standard, combining the envelope condition for current assets and the first-order condition for next period’s assets yields the asset pricing equation

\[
q_t(s^{t+1}) = \beta P(s^{t+1})c(s^t) / P(s^t)c(s^{t+1});
\]

the particular functional form of the pricing kernel arises because of the assumption of logarithmic utility of consumption. Suppressed in the notation for the household’s value function is the fact that the household’s labor income in the production sector depends on the distribution of employment across firms and on the wages paid by each firm.

### 2.5 Wage determination

I assume that wages are set following the assumption on bargaining first made by Stole and Zwiebel (1996a,b), adapted to the economic environment I study in this paper. In order to set up the equation which characterizes the bargained wage, I need to write expressions for the marginal benefit to the firm of employing a marginal worker and to the household of having a marginal member employed at a firm.
2.5.1 Marginal value of employment for the firm

First, consider the firm. Denote by \( \hat{J}_n(n; s^t, \pi^t) \) the marginal benefit to the firm of employing a worker at the beginning of period \( t \), after learning the values of aggregate and idiosyncratic productivity shocks (but before firing and recruitment occur), and by \( J_n(n; s^t, \pi^t) \) the marginal benefit to the firm of employing the worker at the beginning of the production phase of period \( t \) (before production and wage payments occur, but after the recruitment phase is closed). This notation and timing is consistent with that introduced in (4) and (5). Accordingly, I can obtain an expression for \( J_n(n; s^t, \pi^t) \) by differentiating equation (5), to see that

\[
J_n(n; s^t, \pi^t) = \frac{\partial}{\partial n} [p_t \pi_t y(n) - nw(n; s^t, \pi^t)] + (1 - \delta) \sum_{s^{t+1}|s^t} q_t(s^{t+1}) \sum_{\pi^{t+1}} \mu(\pi_t, \pi_{t+1}) \hat{J}_n(n; s^{t+1}, \pi^{t+1}).
\]

(16)

Next, note that if the firm’s value function \( \hat{J}(n; s^t, \pi^t) \) is strictly concave in \( n \), then the optimal hiring and firing behavior of the firm takes a particularly simple form because of the constant marginal costs of recruitment and of firing workers (in fact, the latter cost is zero). There exist two values \( n^*(s^t, \pi^t) < \bar{n}(s^t, \pi^t) \), which I will call respectively the firm’s hiring target and firing target. The firing target is such that any firm which, upon learning the realization of the shocks \( p_t \) and \( \pi_t \), has current employment \( n > \bar{n}(s^t, \pi^t) \), will immediately fire \( n - \bar{n}(s^t, \pi^t) \) workers so as to reduce its employment to \( \bar{n}(s^t, \pi^t) \). Such a firm will not recruit. The hiring target is such that any firm which, upon learning \( p_t \) and \( \pi_t \), finds that its current employment satisfies \( n < n^*(s^t, \pi^t) \) fires no workers and recruits \( n^*(s^t, \pi^t) - n \) workers so as to increase its employment to \( n^*(s^t, \pi^t) \). I therefore assume that the firm’s value function is indeed strictly concave for now, so that these properties hold. Note that the firing target is characterized by the condition that the marginal value to the firm during the production phase of having an additional worker equal zero. Moreover, because any firm that begins period \( t \) with history \( (s^t, \pi^t) \) with more workers than its firing target, \( \bar{n}(s^t, \pi^t) \), immediately fires all the surplus at zero cost, and because any such firm will not hire, we have that for any \( n \geq \bar{n}(s^t, \pi^t) \),

\[
\hat{J}_n(n; s^t, \pi^t) = \hat{J}_n(\bar{n}(s^t, \pi^t); s^t, \pi^t) = J_n(\bar{n}(s^t, \pi^t); s^t, \pi^t) = 0.
\]

(17)

Analogously, the hiring target is characterized by the condition that the marginal value to the firm during the production phase equal the cost of recruiting an additional worker, \( k^r(s^t) \). Any firm which enters period \( t \) and finds, after learning \( p_t \) and \( \pi_t \), that it is below its hiring target \( n^*(s^t, \pi^t) \), immediately recruits the workers it lacks at the same constant marginal cost, so that for any \( n \leq n^*(s^t, \pi^t) \), we have that

\[
\hat{J}_n(n; s^t, \pi^t) = \hat{J}_n(n^*(s^t, \pi^t); s^t, \pi^t) = J_n(n^*(s^t, \pi^t); s^t, \pi^t) = k^r(s^t).
\]

(18)

Finally, any firm whose initial employment lies strictly between the hiring and firing target neither
hires nor fires. It immediately follows that for such a firm,

$$\hat{J}_n(n; s^t, \pi^t) = J_n(n; s^t, \pi^t).$$

(19)

From (17), (18), and (19) it follows that the marginal benefit to the firm from employing a worker is in equilibrium always the same at the beginning of period $t$ and during the hiring phase. It then follows from substitution into (16) that

$$J_n(n; s^t, \pi^t) = \frac{\partial}{\partial n} [p_t \pi_t y(n) - nw(n; s^t, \pi^t)] + (1 - \delta) \sum_{s^{t+1}|s^t} q_t(s^{t+1}) \sum_{\pi^{t+1}} \mu(\pi_t, \pi_{t+1}) J_n(n; s^{t+1}, \pi^{t+1}).$$

(20)

In addition, it is useful to define $\tilde{J}_n(w, n; s^t, \pi^t)$ to be the marginal value to the firm during the production phase of period $t$ of employing a worker who will be paid an arbitrary wage $w$ during period $t$ only, but after period $t$ will, if employment continues, return to having her wage given by the same bargaining protocol as all other workers. That is, during period $t$ alone, this worker is paid $w$ rather than the wage $w(n; s^t, \pi^t)$ that would be selected by the wage bargaining mechanism to be described below. It is immediate that

$$\tilde{J}_n(w, n; s^t, \pi^t) = -(w - w(n; s^t, \pi^t)) + J_n(n; s^t, \pi^t),$$

(21)

since the worker is equivalent to any other worker except for her different wage payment during period $t$.

### 2.5.2 Marginal value of employment for the household

I now turn to consider the marginal value to the household of having a worker employed.\(^6\) Analogously to the notation used for the firm, denote by $\hat{V}_n(n; s^t, \pi^t)$ the marginal value to the household of having a worker employed in a firm with idiosyncratic shock history $\pi^t$ and employment $n$ at the beginning of period $t$, after the aggregate and idiosyncratic productivity shocks have been realized but before firing, recruitment, or production have occurred. Denote by $V_n(n; s^t, \pi^t)$ the marginal value to the household of having a worker employed in such a firm after firing and recruitment are complete during period $t$, but before production and wage payments occur. Denote by $\hat{V}^u(s^t)$ the marginal value to the household of an unemployed worker before the labor market closes in period $t$. Finally, define

$$x(s^t) = \hat{V}^u(s^t) - \beta \sum_{s^{t+1}|s^t} \frac{P(s^{t+1})}{P(s^t)} \hat{V}^u(s^{t+1}).$$

(22)

$x(s^t)$ can be interpreted as the ‘rental value’ of an unemployed worker for a single period. If $\hat{V}^u(\cdot)$ is independent of the aggregate history, then $x(\cdot) = (1 - \beta) \hat{V}^u(\cdot)$. More generally (22) accounts for

---

\(^6\)The measure of workers in the household is of course fixed, so more formally, the values defined in this paragraph are Lagrange multipliers on the appropriate constraints in the household’s problem.
changes over time in the value of an unemployed worker due to changes in job-finding probabilities and in wages of employed workers.

The first observation to make concerning these values is that $\hat{V}_n(n; s^t, \pi^t) = V_n(n; s^t, \pi^t)$. There are two cases to consider. The first is the case when the worker is employed at a firm that enters period $t$ with at most as many workers than its firing target, $\bar{n}(s^t, \pi^t)$. Such a firm may hire other workers (if its employment is less than its hiring target, $n^*(s^t, \pi^t)$), but it does not fire, so that any worker who begins the period employed by the firm continues employed into the production phase for sure. In this case, the claim follows immediately. The second possibility is the case when the firm enters period $t$ with more workers than its firing target. In this case, some workers are fired, so that the household loses the marginal benefit of having them employed at the firm; however, it follows immediately from the condition characterizing the firing target (17), together with the bargaining equation (27) below, that the surplus over unemployment associated with the workers at this firm who are not fired is in fact also equal to zero.

Using this observation, I can therefore write a recursive expression for the marginal value the household from having a worker employed at a firm in state $(s^t, \pi^t)$ and which has $n$ employees after recruitment is completed and before the production stage has commenced. During period $t$, the marginal benefit to the household from having a worker employed at this firm is that the worker receives the wage $w(n; s^t, \pi^t)$, which is valued at the household’s marginal utility of consumption, and the marginal cost is the disutility of labor in the production sector, $\phi^y$. Then, conditional on the firm avoiding exogenous destruction at the end of period $t$, the worker remains employed by this firm for the production phase of period $t+1$ with probability 1 if the firm does not fire workers; if not, then the marginal surplus over unemployment obtained by the household from having a worker employed in such a firm is zero in any case. That is, the equation takes the following form:

$$V_n(n; s^t, \pi^t) = \frac{w(n; s^t, \pi^t)}{c(s^t)} - \phi^y + \beta \delta \sum_{s^{t+1}|s^t} \frac{P(s^{t+1})}{P(s^t)} \hat{V}_u(s^{t+1})$$

$$+ \beta(1-\delta) \sum_{s^{t+1}|s^t} \frac{P(s^{t+1})}{P(s^t)} \sum_{\pi_{t+1}} \mu(\pi_t, \pi_{t+1}) V_n(n'(n; s^{t+1}, \pi_{t+1}); s^{t+1}, \pi^{t+1}); (23)$$

(Recall from (6) that $n'(n; s^{t+1}, \pi^{t+1})$ denotes the firm’s policy for the state-contingent evolution of its employment.) Subtract $\hat{V}_u(s^t)$ from both sides of (23) and rearrange to see that, if I define the net surplus to the household from having an employed worker in period $t$ relative to an unemployed worker by

$$S(n; s^t, \pi^t) = V_n(n; s^t, \pi^t) - \hat{V}_u(s^t), (24)$$

A reader familiar with the analysis of Shimer (2010) might wonder why equation (23) does not explicitly incorporate any terms relating to the probability that an unemployed worker finds a job (compare his equation 2.39); of course, these terms are implicitly present in the value of $\hat{V}_u(\cdot)$. This is because of a difference of timing assumptions between his model and mine; he assumes that a worker who is unemployed at the beginning of a period cannot work in the production sector until the following period, whereas I assume that employment starts already during the current period. I make this assumption because it is more convenient in generating a simple solution for wage bargaining.
\[ S(n; s^t, \pi^t) = \frac{w(n; s^t, \pi^t)}{c(s^t)} - \phi y - \pi(s^t) \]
\[ + \beta(1 - \delta) \sum_{s^{t+1}|s^t} \frac{P(s^{t+1})}{P(s^t)} \sum_{\pi_{t+1}} \mu(\pi_t, \pi_{t+1}) S(n'(n; s^{t+1}, \pi_{t+1}); s^{t+1}, \pi_{t+1}) \]  

(25)

Finally, define \( \tilde{S}(w, n; s^t, \pi^t) \) to be the surplus relative to unemployment for the household of having a worker employed at a firm with idiosyncratic shock history \( \pi^t \) and employment \( n \) if the wage during period \( t \) only is equal to \( w \) and not to the wage \( w(n; s^t, \pi^t) \) that would be selected by the wage bargaining mechanism to be described below. (In periods after period \( t \), it is assumed that the wage returns to the bargained value.) That is,
\[ \tilde{S}(w, n; s^t, \pi^t) = w - w(n; s^t, \pi^t) + S(n; s^t, \pi^t), \]  

(26)

since the household values the alteration in wage during period \( t \) according to its marginal utility of consumption; after period \( t \), the wage returns to the bargained level, so that all other components of the value to the household of such a worker are given by (23).

### 2.5.3 Bargaining Equation and Wages

I assume that firms bargain with all their workers following Stole and Zwiebel (1996a,b). That is, the wage \( w \) to be paid by a firm with \( n \) workers in period \( t \) after history \( s^t \) if the history of idiosyncratic shocks is \( \pi^t \) solves
\[ w(n; s^t, \pi^t) = \arg \max_w \tilde{S}(w, n; s^t, \pi^t)^\eta \tilde{J}_n(w, n; s^t, \pi^t)^{1-\eta}. \]  

(27)

In this equation \( \eta \in (0, 1) \) is the bargaining power of workers, and is a structural parameter of the model. The assumption is intuitively that firms and workers bargain over the marginal surplus generated by their employment relationship; the outside option for the firm is to produce with one fewer worker, while the outside option for the worker is unemployment. The assumption of Stole and Zwiebel is that the firm bargains individually with its entire workforce, treating each as marginal.\(^8\)

---

\(^8\)The attentive reader will notice that interpreting (24) as a surplus over an outside option in the bargaining equation (27) implicitly involves a problematic timing assumption. Should a worker reject the firm’s offer, this formulation assumes that the value \( \tilde{V}_u(s^t) \) is available, which requires that the labor market remains open for her to look for another job. For incumbent workers who were already employed by the firm the previous period, this assumption is plausible if bargaining takes place before the recruitment phase of the period, with the firm and workers committed to employment at the bargained wage during the production phase which follows. For workers newly hired in the current period, the timing assumption is more problematic: it must be that, although hired this period, the worker can return to the labor market and search again for a job this period should bargaining fail. This artificial assumption is necessary to generate a convenient solution for the bargained wage in the discrete-time environment considered here and would not be an issue in continuous time. The theoretical results of the paper would be unchanged in continuous time, but computations would be less tractable in settings with aggregate fluctuations.
From (21) it follows that \( \tilde{J}_n(w;n; s^t, \pi^t), n; s^t, \pi^t) = J_n(n; s^t, \pi^t) \) and \( \frac{\partial}{\partial w} \tilde{J}_n(w;n; s^t, \pi^t), n; s^t, \pi^t) = -1. \) From (26) it follows that \( \tilde{S}(w(n; s^t, \pi^t), n; s^t, \pi^t) = S(n; s^t, \pi^t) \) and \( \frac{\partial}{\partial w} \tilde{S}(w(n; s^t, \pi^t), n; s^t, \pi^t) = \frac{1}{c(\sigma)}. \) It follows that the solution to (27) is such that

\[
\eta J_n(w(n; s^t, \pi^t), n; s^t, \pi^t) = (1 - \eta)S(n; s^t, \pi^t), n; s^t, \pi^t)c(s^t). \tag{28}
\]

Equation (28) holds for all \((n, s^t, \pi^t). \) Substituting into (20) and (23) and using the asset pricing equation (15) establishes immediately that

\[
(1 - \eta) [w(n; s^t, \pi^t) - (\phi y + x(s^t))c(s^t)] = \eta \frac{\partial}{\partial n} \left[ p_t \pi_t y(n) - nw(n; s^t, \pi^t) \right]. \tag{29}
\]

The power of equation (29) is that it establishes that wages depend on aggregate conditions only via the consumption of the representative household, \( c(s^t), \) the value of an unemployed worker, \( x(s^t), \) and the current aggregate productivity shock, \( p_t. \) Equation (29) can be solved through the use of the appropriate integrating factor; the general solution is

\[
w(n; s^t, \pi^t) = (1 - \eta)(\phi y + x(s^t))c(s^t) + n^{-\frac{1}{\eta}} \left[ C + p_t \pi_t \int_0^n y^{\frac{1}{\eta}} y'(\nu) d\nu \right],
\]

where \( C \) is a constant of integration. However, assuming that the total wage bill of a firm remains finite as its employment level \( n \) becomes small requires that \( C = 0. \) I record this result as the following Lemma.

**Lemma 1.** The wage paid by a production firm with \( n \) workers and idiosyncratic shock history \( \pi^t \) in period \( t \) after history \( s^t \) is given by

\[
w(n; s^t, \pi^t) = (1 - \eta)(\phi y + x(s^t))c(s^t) + p_t \pi_t n^{-\frac{1}{\eta}} \int_0^n y^{\frac{1}{\eta}} y'(\nu) d\nu. \tag{30}
\]

I assume henceforth that the value of \( \eta \) and the form of the production function are such that the integral on the right side of (30) is finite, with

\[
\lim_{n \to 0^+} \int_0^n n^{\frac{n-1}{\eta}} \int_0^n \nu^{\frac{1}{\eta}} y'(\nu) d\nu = 0. \tag{31}
\]

In the case that \( y(n; p, \pi) = p \pi n^\alpha \) is Cobb-Douglas, a necessary and sufficient condition to ensure this is that \( \frac{1}{\eta} + \alpha > 1, \) so that \( \nu^{\frac{1}{\eta}} y'(\nu) = A \nu^{\frac{1}{\eta} - \alpha} \) is integrable near \( \nu = 0. \) Since \( \eta < 1 \) and \( \alpha > 0, \) the condition is always satisfied in this case, which will be the case used in the calibrated model. In this case the integral on the right side of (30) can be calculated in closed form, establishing that

\[
w(n; s^t, \pi^t) = (1 - \eta)(\phi y + x(s^t))c(s^t) + \frac{\alpha \eta}{\alpha \eta + 1 - \eta} p_t \pi_t n^{\alpha - 1}. \tag{32}
\]

More generally, an intuition for the form of the wage function defined by (30) can be gained by
rewriting the equation as

\[ w(n; s^t, \pi^t) = (1 - \eta)(\phi^y + x(s^t))c(s^t) + \eta \frac{\int_0^n \nu^{1-\eta} p_t \pi_t y'(\nu) d\nu}{\int_0^n \nu^{1-\eta} d\nu}. \]

That is, wages are a weighted average of two terms. The first term is the disutility of labor (measured in units of consumption good by adjusting using the marginal utility of consumption of the representative household). The second term is itself a weighted average of the inframarginal products of workers at smaller firms than \( n \). Stole and Zwiebel (1996a) give an account of how such a term can arise in a static model; intuitively the reason that inframarginal products arise are that they are relevant for the firm’s outside option, which is that it bargains with a smaller workforce if the marginal worker were to refuse its wage offer.

It is useful to record that the functional form for wages given by (30) guarantees that the the firm’s flow profit function,

\[ y(n; p, \pi) - nw(n; s^t, \pi^t) \]

has some convenient properties. Using (30) it can be seen that flow profit satisfies

\[ y(n; s^t, \pi^t) = p_t \pi_t \tilde{y}(n) - (1 - \eta)n(\phi^y + x(s^t))c(s^t) \quad (33) \]

The following lemma establishes that \( \tilde{y}(n) \) itself has the standard properties of a neoclassical production function, which justifies assumptions made above in deriving the nature of optimal firing and recruitment policies for production firms.

**Lemma 2.** The function

\[ \tilde{y}(n) = y(n) - n \frac{\eta^{n-1}}{\eta} \int_0^n \nu^{\frac{1-n}{\eta}} y'(\nu) d\nu \quad (34) \]

is always nonnegative, strictly increasing, strictly concave and satisfies the Inada conditions \( \lim_{n \to 0} \tilde{y}'(n) = +\infty \) and \( \lim_{n \to \infty} \tilde{y}'(n) = 0 \).

The proof of Lemma 2 is in the Appendix.

It is also useful to note that the bargaining equation provides a particularly simple characterization of \( x(s^t) \), the rental value of an unemployed worker. The Bellman equation characterizing the value of an unemployed worker can be written as

\[ \hat{V}^u(s^t) = f(\tilde{\theta})EV_n(n; s^t, \pi^t) + (1 - f(\tilde{\theta}))\beta\hat{V}^u(s^{t+1}) \quad (35) \]

where the expectation is taken over all hiring firms in proportion to their recruitment purchases. However, at firm which recruits a nonzero number of workers, (28) together with the characterization of optimal hiring in Section 2.5.1 implies that

\[ V_n(n; s^t, \pi^t) = \hat{V}^u(s^t) + \frac{\eta}{1 - \eta} \frac{J_n(n; s^t, \pi^t)}{c(s^t)} = \hat{V}^u(s^t) + \frac{\eta}{1 - \eta} k^r(s^t). \]
(9) and (14) imply that \(k^r(s^t) = \gamma \phi^r c(s^t)/m(\theta(s^t))\), so that in fact

\[
V_n(n; s^t, \pi^t) = \tilde{V}^u(s^t) + \frac{\eta \gamma \phi^r}{(1 - \eta)m(\theta(s^t))}.
\]

It follows from substituting into (35) and using the relation \(f(\theta) = \theta m(\theta)\) that

\[
x(s^t) = \tilde{V}^u(s^t) - \beta E\tilde{V}^u(s^{t+1}) = \frac{\theta(s^t) \eta \gamma \phi^r}{(1 - f(\theta(s^t)))(1 - \eta)}.
\]  

(36)

2.6 Market Clearing

To close the model requires imposing several market clearing conditions. Two of these have already been mentioned: an interior equilibrium in the market for labor in the construction sector requires that the wage satisfy (13), and an interior equilibrium in the market for labor in the recruitment sector requires that the wage satisfy (14). In addition, I assume that the market for the consumption good clears, so that the representative household’s consumption equals production by firms, that is,

\[
c(s^t) = \sum_{n} \sum_{\pi^t} \int_0^\infty p_{n\pi} y(n) dG(n; s^t, \pi^t),
\]  

(37)

I need to impose several consistency conditions. First, the evolution of the firm size distribution must be consistent with the firing and recruitment decisions of firms, so that for incumbent firms

\[
G(n; s^t, \pi^t) = (1 - \delta) \int_{T(n; s^t, \pi^t)} dG(\nu; s^{t-1}, \pi^{t-1})
\]

(38)

where \(T(n; s^t, \pi^t) = \{\nu \mid n \geq \nu - f(\nu; s^t, \pi^t) + h(\nu; s^t, \pi^t)\}\) is the set of employment values at period \(t - 1\) that would lead the firm, conditional on the realization of productivity shocks for period \(t\), to fire and/or recruit so that its employment is at most \(n\) during the production stage of period \(t\). For entrants,

\[
G(n; s^t, (0, \ldots, 0, \zeta_j)) = \begin{cases} 
0 & n < n^e(0; s^t, (0, \ldots, 0, \zeta_j)) \\
n^e(s^t) F_{\pi}^e(\zeta_j) & \text{otherwise}. 
\end{cases}
\]

(39)

Here \(n^e(s^t)\) is the measure of entrant firms following history \(s^t\).

Second, the ratio of recruiters hired by recruitment firms to unemployed workers must equal \(\theta(s^t)\); that is,

\[
\theta(s^t) = \frac{c^e(s^t)}{u(s^t)},
\]

(40)

where \(u(s^t)\), the number of unemployed workers after the firing stage of period \(t\), evolves according
The explanation of this equation is that unemployed workers arise from three sources, those who were unemployed in the previous period and failed to find a job in the production sector (which occurred with probability $1 - f(\theta(s^{t-1}))$), those who were employed in the production sector during the previous period but whose firm was exogenously destroyed (with probability $\delta$), and those who were fired from the production sector at the beginning of the current period after their firm observed the current aggregate and idiosyncratic productivity shocks.

Last, employment in the construction sector must be consistent with construction firms’ output of factories, which in turn must equal entrant firms’ demand,

$$e^f(s^t) = \kappa n^e(s^t),$$

(42)

employment in the recruitment sector must be consistent with recruitment firms’ production of recruitment opportunities, which in turn must be consistent with the hiring policies of production firms,

$$\frac{1}{\gamma} e^r(s^t) m(\theta(s^t)) = (1 - \delta) \sum_{\pi_t} h(n; s^t, \pi^t) \int dG(n; s^{t-1}, \pi^{t-1}),$$

(43)

and labor supply by the household to the production sector must be consistent with employment at active firms,

$$e^u(s^t) = \sum_{s^t} \sum_{\pi^t} \int_0^\infty n \, dG(n; s^t, \pi^t).$$

(44)

### 2.7 Equilibrium

I am now in a position to make the following definition of equilibrium.

**Definition 1.** An *equilibrium* is a set of (aggregate history-dependent) sequences for consumption $c(s^t)$, asset holding $a(s^t)$, labor supply in the construction, recruitment, and production sectors $e^f(s^t)$, $e^r(s^t)$, and $e^u(s^t)$, unemployment $u(s^t)$, the recruiter-unemployment ratio $\theta(s^t)$, entry by new production firm $n^e(s^t)$, wages in the construction and recruitment sectors $w^f(s^t)$ and $w^r(s^t)$, the prices of a factory and of a recruitment opportunity $k^f(s^t)$ and $k^r(s^t)$, and the date-0 prices of history-$s^t$ consumption goods $q_0(s^t)$ together with a set of (aggregate and idiosyncratic history-dependent) sequences for firing, recruitment, and target employment by production firms $f(n; s^t, \pi^t)$, $h(n; s^t, \pi^t)$, and $n'(n; s^t, \pi^t)$, wages paid by production firms $w(n; s^t, \pi)$, and the distribution of production firms by employment level $G(n; s^t, \pi^t)$, such that

- the household chooses consumption, savings, and labor supply to the construction and recruitment sectors to maximize the right side of (11), so that (13), (14), and (15) hold;
• the household’s budget constraint (12) holds;
• construction firms choose factory production and recruitment firms choose recruitment opportunity production optimally subject to perfect competition, so that (8) and (9) hold;
• wages in the production sector are determined by (30), and firing and recruitment are chosen to maximize profits, so that \( f(n; s^t, \pi^t) \) and \( h(n; s^t, \pi^t) \) maximize the expression on the right side of (4);
• the free entry condition (7) holds, and entry is always non-negative, and strictly positive only if (7) holds with equality;
• markets clear for consumption goods, factories, and recruitment opportunities, so that (37), (42), and (43) hold;
• labor supply by the household to the production sector is consistent with labor demand by production firms, so that (44) holds;
• unemployment evolves according to (41);
• individual production firm sizes evolve according to (3) and the firm size distribution evolves according to (38) and (39); and
• \( \theta(s^t) \) is consistent with the recruitment activities of firms and the number of unemployed workers, according to (40).

The definition of equilibrium looks somewhat unwieldy, since the interaction between aggregate variables and the idiosyncratic shocks process for the productivity of production firms looks like it must make solving for a dynamic equilibrium intractable. Fortunately, because of the balanced growth preferences of households, and provided that the initial conditions of the economy are appropriate, in fact an equilibrium exists with a very simple structure, that unemployment, labor supply to the construction and recruitment sectors and employment in the production sector, the entry of production firms, and the distribution of production firm sizes are constant over time and independent of the history of aggregate productivity \( p_t \).

**Definition 2.** A *stationary equilibrium* is an equilibrium with the property that there exist constant values \( \bar{c}, \bar{a}, \bar{e}^f, \bar{e}^r, \bar{e}^y, \bar{u}, \bar{n}^e, \bar{w}^f, \bar{w}^r, \bar{k}^f, \bar{k}^r \), and functions \( \bar{f}, \bar{h}, \bar{n}', \bar{w}, \bar{G} : [0, \infty) \times \{\zeta_1, \ldots, \zeta_{m_p}\} \rightarrow [0, \infty) \), with \( G(\cdot, \zeta_j) \) nondecreasing for \( 1 \leq j \leq m_p \), such that for all \( t \) and all \( s^t \),

- \( a(s^t) = \bar{a}, \ e^f(s^t) = \bar{e}^f, \ e^r(s^t) = \bar{e}^r, \ e^y(s^t) = \bar{e}^y, \ u(s^t) = \bar{u}, \ n^e(s^t) = \bar{n}^e; \)
- \( c(s^t) = p_t \bar{c}, \ w^f(s^t) = \bar{w}^f, \ w^r(s^t) = \bar{w}^r, \ k^f(s^t) = \bar{k}^f, \ k^r(s^t) = \bar{k}^r; \)
- for all \( n \) and all \( \pi^t, f(n; s^t, \pi^t) = \bar{f}(n, \pi_t), \ h(n; s^t, \pi^t) = \bar{h}(n, \pi_t), \ n'(n; s^t, \pi^t) = \bar{n}'(n, \pi_t), \) and \( G(n; s^t, \pi_t) = \bar{G}(n, \pi_t); \) and
• \( w(n; s^t, \pi^t) = p_tw(n, \pi_t) \).

**Proposition 1.** A stationary equilibrium exists.

The proof of Proposition 1 is in the Appendix. The structure of the proof is as follows. The first step is to normalize the value function for firms by dividing through by current aggregate productivity. Under the assumption that the equilibrium is stationary, the normalized value function is then independent of the aggregate history because of the functional form of wages given by (30). This means that the firm’s normalized value depends only on current idiosyncratic productivity and current employment. Market clearing in the construction and recruitment sectors then shows that the cost of a factory and of recruiting a worker depend only on the level of consumption, \( \bar{c} \), and on the recruiter-unemployment ratio \( \bar{\theta} \). Given \( \bar{c} \), there is a unique \( \bar{\theta} \) consistent with free entry; then by continuity I can also find a value for \( \bar{c} \) that is consistent with goods market clearing. The construction of the remainder of the equilibrium allocation is then standard.

The existence of a stationary equilibrium is possible for the same reasons as in Shimer (2010), since I assume balanced growth preferences and that the entry and recruiting sectors are labor-intensive. Therefore, the income and substitution effects exactly offset. When aggregate productivity is temporarily high, all else equal, active production firms would increase recruitment and inactive production firms would increase entry. However, the representative household is made richer by the positive aggregate shock, which it benefits from via an increase in wages in all three sectors of the economy (although not from dividend payments, since in a stationary equilibrium, the dividend payment \( d(s^t) \equiv 0 \) as payment of entry costs for new firms always exactly offset profits from active firms). This increases its desire to consume today, which, in general equilibrium, must lead to an increase in the interest rate. When the utility function of the household takes the balanced-growth form indicated by (41), these income and substitution effects exactly cancel each other, making a stationary equilibrium possible.

The most noteworthy implication of Proposition 1 is that a stationary equilibrium exists independently of the shock process for idiosyncratic productivity. That is, if the model of this section is a good description of the world, then essentially any observations on the firm size distribution and on the stochastic process for the size of an individual firm can be rationalized by the model using an appropriate choice for the stochastic process for idiosyncratic productivity, \( \mu \). (Intuitively, suppose that idiosyncratic productivity includes both a permanent and a transitory component. The variance of the transitory component can be chosen to match the cross-sectional dispersion in firm employment growth, and then conditionally on this, the variance of the permanent component can be chosen to match the cross-sectional dispersion in firm employment levels. In the quantitative analysis in Section 4 I perform such a calibration.) Such data only have implications for the dynamics of aggregate variables if either preferences are not consistent with balanced growth (so that income and substitution effects do not cancel), or if the entry and recruitment technologies do not, as here, use technologies that require only labor as an input. Therefore, in the next section, I modify the model in two ways that are common in the search literature. There, I assume that workers are risk-neutral, so that the general equilibrium effect of productivity on interest rates is
absent, so that an increase in productivity no longer drives up the disutility of labor, measured
terms of consumption goods. I assume also that the entry and recruitment technologies use the
consumption good as an input.

2.8 Constructive Solution Algorithm

The proof of Proposition 1 is not constructive, so it is useful to provide an algorithm for constructing
the equilibrium allocation. In the proof, I show that the normalized Bellman equation for a firm
in a stationary allocation can be written

$$\hat{K}(n, \pi) = \max_{n'} \left[ -\tilde{k}^r \max(n' - n, 0) + \pi \tilde{y}(n') - n'(1 - \eta)(\phi \bar{x})\bar{c} + \beta(1 - \delta) \sum_{\pi_{t+1}} \mu(\pi_t, \pi_{t+1}) \hat{K}(n', \pi_{t+1}) \right],$$

(45)

where $$\tilde{k}^r = \gamma \phi \bar{c}/m(\theta)$$, $$\bar{x} = x(s^t)$$ is defined in (22) and is constant in a stationary allocation,
and $$\tilde{y}(\cdot)$$ is defined in (34). Next, recall that the firing and hiring policies of production firms are
characterized by firing and hiring targets, so that any firm which finds, after learning the shocks in
period $$t$$, that its employment level exceeds its firing target, fires all its excess workers above that
target; likewise, a firm with fewer workers than its firing target immediately hires to reach that level.
The first-order conditions characterizing these targets given by (17) and (18) immediately simplify
in a stationary allocation to establish that the firing and hiring targets are both independent of
the history $$s^t$$ and of the history of idiosyncratic shocks $$\pi^t$$, except through a dependence on the
current-period value $$\pi_t$$. The first-order conditions characterizing the two targets, which I now
denote simply as $$n^*(\pi)$$ and $$\tilde{n}(\pi)$$, are

$$\hat{K}_n(n^*(\pi), \pi) = K_n(n^*(\pi), \pi) = \tilde{k}^r \quad \text{and} \quad \hat{K}_n(\tilde{n}(\pi), \pi) = K_n(\tilde{n}(\pi), \pi) = 0.$$ \hspace{1cm} (46)

Then for $$n < n^*(\pi),$$

$$\hat{K}(n, \pi) = -(n^*(\pi) - n)\tilde{k}^r + K(n^*(\pi), \pi) \quad \text{and} \quad \hat{K}_n(n, \pi) = \tilde{k}^r,$$ \hspace{1cm} (47)

and for $$n > \tilde{n}(\pi),$$

$$\hat{K}(n, \pi) = K(\tilde{n}(\pi), \pi) \quad \text{and} \quad \hat{K}_n(n, \pi) = 0.$$ \hspace{1cm} (48)

Finally, for $$n \in [n^*(\pi), \tilde{n}(\pi)]$$, of course, $$K(n, \pi) = K(n, \pi)$$, since neither firing nor recruitment
occurs. Thus in all cases, $$\hat{K}_n(n, \pi) = K_n(n, \pi)$$. In summary, (45) can be rewritten as

$$K(n, \pi) = \begin{cases} -(n^*(\pi) - n)\tilde{k}^r + K(n^*(\pi), \pi) & n \leq n^*(\pi) \\ \tilde{y}(n) - n(1 - \eta)(\phi \bar{x})\bar{c} + \beta(1 - \delta) \sum_{\pi'} \mu(\pi, \pi')K(n, \pi') & n^*(p) \leq n \leq \tilde{n}(\pi) \\ K(\tilde{n}(\pi), \pi) & n \geq \tilde{n}(\pi). \end{cases}$$ \hspace{1cm} (49)
Differentiate (49) with respect to $n$ and use this fact to observe that for $n \in [n^*(\pi), \bar{n}(\pi)]$,

$$K_n(n, \pi) = \begin{cases} \hat{k}^r & n \leq n^*(\pi) \\ \hat{g}'(n) - (1 - \eta)(\phi y + \bar{x})\bar{c} + \beta(1 - \delta)\sum_{\pi'} \mu(\pi, \pi')K_n(n, \pi') & n^*(\pi) < n \leq \bar{n}(\pi) \\ 0 & n \geq \bar{n}(\pi). \end{cases}$$

Write $K'_j(n^*_l) = K_n(n^*(\zeta_l), \zeta_j)$ and $K'_j(\bar{n}_l) = K_n(\bar{n}(\zeta_l), \zeta_j)$. Then substituting appropriately into (50) gives $2m^2_y + 2m_\pi$ linear equations in the $2m^2_y + 2m_\pi$ variables \(\{K'_j(n^*_l)\}_{jl} \cup \{K'_j(\bar{n}_l)\}_{jl}\) \(\cup \{\hat{g}'(n^*_l)\}_{l} \cup \{\hat{g}'(\bar{n}_l)\}_{l}\). (The additional $2m_\pi$ equations come because for every $j$, both the first and second cases in (50) hold when $n = n^*(\zeta_j)$, and both the second and third cases hold when $n = \bar{n}(\zeta_j)$.) Note that the form of the system of linear equations depends only on the key endogenous variables $\bar{c}$ and $\bar{\theta}$ (which enter the value of $\hat{k}^r$) and on the ordering of the set $L \equiv \{n^*_l\}_{jl} \cup \{\bar{n}_l\}_{jl}$, since this ordering determines which of the three cases applies in (50). However, conditional on a conjecture for the ordering, a solution to the equations determines \(\{\hat{g}'(n^*_l)\}_{l} \cup \{\hat{g}'(\bar{n}_l)\}_{l}\); since $\hat{g}'$ is strictly decreasing according to Lemma 2, this then determines \(\{n^*_l\}_{l} \cup \{\bar{n}_l\}_{l}\).

The solution to (49) and (50) for \(\{n^*_l\}_{l} \cup \{\bar{n}_l\}_{l}\) is unique conditional on the rank order of the elements of $L$ and the value of $\bar{c}$. It is instructive to see why, using the structure of the equations (49) and (50). First suppose that for some $l$, the value of $n^*_l$ is known and consider the set of $m_\pi$ equations given by (50) for $j = 1, 2, \ldots, m_\pi$ with $n = n^*_l$, and assuming that the second case always applies (so that a firm with $n^*_l$ workers never hires or fires). The coefficient matrix for this set of linear equations in \(\{K'_j(n^*_l)\}_{j}\) is invertible. It also implies that for each $j$ and each $j'$, $\hat{K}'_{j'}(n^*_l)$ is increasing in $\pi_j \hat{g}'(n^*_l) - (1 - \eta)(\phi y + \bar{x})\bar{c}$, and therefore strictly decreasing in $n^*_l$. For any $j$ for which $n^*_j \not\in [n^*_j, \bar{n}_j]$, then replace the equation for $K'_j(n^*_l)$ assuming no hiring and firing with the correct equation arising from (50). This equation is of the form $\hat{K}'_{j'}(n^*_l) = s$ for some constant $s$, and so adding it does not affect the full rank of the coefficient matrix, nor the property that $S_{j'}(n^*_l)$ is strictly decreasing in $n^*_l$ for any $j'$ such that $n^*_l \in [n^*_j, \bar{n}_j']$. Repeating this step as many times as necessary establishes that there is a unique solution for \(\{K'_j(n^*_l)\}_{j}\), with the same comparative static property in $n^*_l$ just mentioned. In particular, note that the value of $\hat{K}'_{j'}(n^*_l)$ is decreasing in $n^*_l$.

Also, note that this solution depends only on the rank order of $L$ and the value of $n^*_l$. One can then proceed to use (50) to obtain the additional equation, so far unused, that $K'_j(n^*_l) = (1 - \eta)(\phi y + \bar{x})\bar{c}$, which is independent of $n^*_l$; thus it follows that there is also a unique solution for $n^*_l$. A similar argument then applies to establish that there is also a unique solution for \(\{K'_j(\bar{n}_l)\}_{j}\) and for $\bar{n}_l$. Repeating this argument for all $l$ establishes the claimed uniqueness property.

I then need to verify that this is consistent with the guessed ordering. If so, then substituting into (45) gives another system of $2m^2_y$ linear equations in \(\{K'_j(n^*_l)\}_{jl} \cup \{K'_j(\bar{n}_l)\}_{jl}\). These must be consistent with the free entry condition, which takes the form

$$\sum_{j=1}^{m_\pi} F^e_{\pi}(\zeta_j) \hat{K}(0, \zeta_j) = \hat{k}^f.$$ (51)
This suggests the following algorithm for constructing a solution.

1. Guess values for $\bar{c}$ and $\bar{\theta}$.

2. Guess the rank order of the elements of $L$.

3. Solve the $2m^2 + 2m_\pi$ linear equations given by (50) for $\left\{K_j'(n^*_i)\right\}_{jl} \cup \left\{K_j'(\bar{n}_i)\right\}_{jl} \cup \left\{\bar{y}'(n^*_i)\right\}_l \cup \left\{\bar{y}'(\bar{n}_i)\right\}_l$.

4. Solve for $\left\{n^*_i\right\}_l \cup \left\{\bar{n}_i\right\}_l$. Verify whether the order of the solution is consistent with the guessed order from step 2. If not, guess a new order and return to step 3.

5. Solve the $2m^2$ equations given by (45) for $\left\{K_j(n^*_i)\right\}_{jl} \cup \left\{K_j(\bar{n}_i)\right\}_{jl}$.

6. Verify whether the free entry condition (51) holds with equality for every $i$. If not, adjust the guess for $\bar{\theta}$ and return to step 2.

7. Verify whether the goods market clears; if not, adjust the guess for $\bar{c}$ and return to step 1.

The only part of the algorithm that is not fully specified above is how to guess the rank order of the set $L$ at step 2; since it consists of $2m^2 + 2m_\pi$ elements, there are many possible orderings available (in fact, $(2m_\pi)!$, which already exceeds $3.6 \times 10^6$ if $m_\pi = 5$). Some can be ruled out since we know, for example, that $n^*_j < \bar{n}_j$ for each $j$, and, if the distribution of a firm’s future idiosyncratic productivity is increasing in the sense of first-order stochastic dominance in today’s productivity, we also know that $n^*_j < n^*_k$ and $\bar{n}_j < \bar{n}_k$ whenever $j < k$. However, this still leaves a large set of possible orderings for $L$. I use the following procedure to generate the correct order. The first time I reach step 2, I use an arbitrary ordering. Next, each time I need to make a new guess for the ordering at step 4, I simply use the ordering implied by the solution for $\left\{n^*_i\right\}_l \cup \left\{\bar{n}_i\right\}_l$ just generated using the old guess. In practice this process quickly converges. Next, the dependence on $\bar{c}$ and $\bar{\theta}$ is smooth, so that standard numerical algorithms for solution-finding work well in step 6 and step 7.

3 Model with Linear Utility

The equilibrium of the model described in the previous section does not generate any aggregate fluctuations. In this, it matches the findings of Shimer (2010) for an environment in which firms wish to hire only a single worker. However, the most familiar version of the benchmark MP model differs in two important respects from the environment studied in the previous section: in the search literature, it is more common to assume that workers have linear preferences over consumption (so that there are no general equilibrium effects on the interest rate to offset a temporary increase in productivity) and that the technologies for creating new firms and recruiting workers use the consumption good as inputs, rather than labor. In this section I modify the model of the previous section by returning to an environment with these features. I show that even though the equilibrium
does now exhibit aggregate fluctuations in employment, unemployment, and vacancy creation, these fluctuations need not be persistent. I show that if aggregate shocks are sufficiently small, then I can use a modification of the algorithm described in Section 2.8 to construct an equilibrium in which labor market tightness is a jump variable, as in the usual equilibrium of the benchmark MP model.9

3.1 Model Description

Since the economic environment is similar to that described in Section 2, for the sake of brevity I here indicate only what is different. Anything not mentioned is unchanged from the earlier model. In particular, the problem of production firms is unchanged.

First, I assume that the technologies of construction and recruitment firms now require as input the consumption good, and not labor, to build a factory or generate a recruitment opportunity. I assume that it takes \( \kappa \) units of the consumption good to produce a factory. Since the construction sector is perfectly competitive, this will also be the price of a factory. I also assume that inputs to the matching technology operated by recruitment firms are now vacancy advertising and unemployed workers. A recruitment firm which uses \( v \) units of advertising generates \( vm(\theta) \) matches with unemployed workers, where the assumptions on \( m(\theta) \) remain as in Section 2.3. A unit of advertising is generated at a cost of \( \gamma \) units of the consumption good. Because the recruitment sector is perfectly competitive, the price to a production firm to recruit a worker is \( k^r = \gamma/m(\theta) \) units of the consumption good.

Next, I assume that workers have linear utility over consumption, so that (41) is replaced by the assumption that the felicity function is of the form

\[ c - e^{\gamma \phi^y}. \] (52)

(Terms for labor supply in the construction and recruitment sectors are now absent since there is no demand for labor in these sectors.) Since the worker is risk-neutral, there is now no role for consumption insurance within the household; therefore it is sufficient to assume that the worker always consumes her income. Under this assumption, households make no economic decisions, since employment in the production sector is determined by the recruitment activities of production firms and not by the decisions of workers. This is just as in the MP model. Note that in this environment the asset pricing equation for the price at date \( t \) of history-\( s^{t+1} \) consumption goods is the risk-neutral equivalent to (15), that is,

\[ q_t(s^{t+1}) = \beta P(s^{t+1}) P(s^t). \] (53)

Wage bargaining is still determined by (27), but the resulting form of the wage equation is slightly different. The firm’s problem is unchanged, so that its marginal surplus \( J_n(n; s^t, \pi^t) \) from employing a worker at the equilibrium wage \( w(n; s^t, \pi^t) \) still satisfies (16), and the marginal value

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9In the MP model, this equilibrium is unique in the continuous-time version of the model, although other equilibria can exist in discrete time (Pissarides, 2000; Shimer, 2005).
from employing a worker whose current-period wage is \( w \) and not \( w(n; s^t, \pi^t) \) still satisfied (21). However, the marginal surplus over unemployment of employment at the equilibrium wage to the household is now given by a version of (25) modified to account for the linear utility of consumption,

\[
S(n; s^t, \pi^t) = w(n; s^t, \pi^t) - \phi y - x(s^t)
\]

\[+ \beta(1 - \delta) \sum_{s^{t+1} | s^t} \frac{P(s^{t+1})}{P(s^t)} \sum_{\pi_{t+1}} \mu(\pi_t, \pi_{t+1}) S(n'(n'; s^{t+1}, \pi^{t+1}); s^{t+1}, \pi^{t+1})
\]

(54)

and the marginal value of employment at an arbitrary wage \( w \) is the analogous modification of (26), namely

\[
\tilde{S}(w, n; s^t, \pi^t) = w - w(n; s^t, \pi^t) + S(n; s^t, \pi^t).
\]

(55)

The first-order condition for the solution to (27) therefore takes the form that

\[
\eta J_n(w(n; s^t, \pi^t), n; s^t, \pi^t) = (1 - \eta) S(w(n; s^t, \pi^t), n; s^t, \pi^t),
\]

(56)

or

\[
(1 - \eta) \left[ w(n; s^t, \pi^t) - \phi y - x(s^t) \right] = \eta \frac{\partial}{\partial n} \left[ p_t \pi_t y(n) - nw(n; s^t, \pi^t) \right].
\]

(57)

As in the proof of Lemma 1, (57) can be solved uniquely for wages, so that the following variant of Lemma 1 holds.

**Lemma 3.** The wage paid by a production firm in the linear utility model with \( n \) workers and idiosyncratic shock history \( \pi^t \) in period \( t \) after history \( s^t \) is given by

\[
w(n; s^t, \pi^t) = (1 - \eta)(\phi y + x(s^t)) + p_t \pi_t n^{-\frac{1}{\eta}} \int_0^n \nu^{\frac{1-\eta}{\eta}} y'(\nu) d\nu.
\]

(58)

Wages now depend on aggregate conditions only through \( x(s^t) \), the rental value of an unemployed worker, and \( p_t \), the value of the current aggregate productivity shock, which is in a sense even simpler than the characterization in Lemma 1, since the consumption of the representative household is no longer important.

### 3.2 Equilibrium

I now turn to characterizing equilibrium in the model with linear utility.

The definition of an equilibrium in this environment is the obvious modification of Definition 1 according to the modifications indicated in the preceding subsection (specifically, that (52), (53), and (58) replace their respective counterparts (41), (15), and (30) from the model with balanced growth preferences); I therefore omit it.

Since income and substitution effects no longer cancel in the setting of the current section, no stationary equilibrium can now exist. A natural alternative conjecture for the structure of an equilibrium is that, analogously to results of Pissarides (2000) and Shimer (2005) for the MP model,
labor market tightness $\theta_t$ may be a jump variable whose value depends only on the current value of aggregate productivity, $p_t$. In this section I investigate whether an equilibrium of this form exists.

For the sake of concreteness, it is convenient to assume that the stochastic process for the aggregate productivity shock is a first-order Markov process, supported on the $m_p$ values $z_1, \ldots, z_{m_p}$. Write $\lambda(p_t, p_{t+1})$ for the probability that the period-$t+1$ aggregate productivity shock is $p_{t+1}$, conditional on the value of the period-$t$ shock $p_t$. If $\theta$ depends only on $p$, then for each $i = 1, \ldots, m_p$ write $\theta_i$ for the value of $\theta$ associated with the productivity shock $\zeta_i$.

If market tightness depends only on the aggregate productivity shock, then (36) shows that $x(s^t)$ depends only on the aggregate productivity shock also; I can therefore write $x_i$ for the value of $x(s^t)$ when $p_t = z_i$. Also $k_r = \gamma/m(\theta)$, so I can likewise write $k_i^r$ when $p_t = z_i$. (58) now shows that wages depend only on a firm’s current employment, current idiosyncratic productivity shock, and on the current aggregate shock. Denote by $J_{ij}(n)$ the Bellman value of a firm with $n$ employees when aggregate productivity is $p_i$ and idiosyncratic productivity is $\pi_j$. The Bellman equation for a firm can be written analogously to (45), as

$$J_{ij}(n) = \max_{n'} \left[ -k_i^r \max(n' - n, 0) + p_i \pi_j \tilde{y}(n') - n'(1 - \eta)(\phi^y + \bar{\pi}_i) + \beta(1 - \delta) \sum_{i' = 1}^{m_p} \sum_{j' = 1}^{m_p} \lambda(z_i, z_{i'}) \mu(\pi_j, \pi_{j'}) J_{i' j'}(n') \right].$$

(59)

The associated Bellman operator is a contraction, so that there is a unique solution (conditional on the values of $(\theta_1, \ldots, \theta_{m_p})$) for $J(\cdot)$. Arguing as in Section 2.8, the firm’s hiring and firing behavior are characterized by hiring targets $n_{ij}^*$ and firing targets $\bar{n}_{ij}$ that depend only on the current aggregate and idiosyncratic productivity states. I can write the analog of (49), that

$$J_{ij}(n) = \begin{cases} 
-(n_{ij}^* - n)k_i^r + J_{ij}(n_{ij}^*) & n \leq n_{ij}^* \\
p_i \pi_j \tilde{y}(n) - n(1 - \eta)(\phi^y + \bar{\pi}_i) + \beta(1 - \delta) \sum_{i' = 1}^{m_p} \sum_{j' = 1}^{m_p} \lambda(z_i, z_{i'}) \mu(\pi_j, \pi_{j'}) J_{i' j'}(n) & n_{ij}^* \leq n \leq \bar{n}_{ij} \\
J_{ij}(\bar{n}_{ij}) & n \geq \bar{n}_{ij}.
\end{cases}$$

(60)

Differentiating produces the analog of (50),

$$J'_{ij}(n) = \begin{cases} 
k_i^r & n \leq n_{ij}^* \\
p_i \pi_j \tilde{y}'(n) - (1 - \eta)(\phi^y + \bar{\pi}_i) + \beta(1 - \delta) \sum_{i' = 1}^{m_p} \sum_{j' = 1}^{m_p} \lambda(z_i, z_{i'}) \mu(\pi_j, \pi_{j'}) J'_{i' j'}(n) & n_{ij}^* \leq n \leq \bar{n}_{ij} \\
0 & n \geq \bar{n}_{ij}.
\end{cases}$$

(61)

Arguing as in the proof of Proposition 1, I can readily establish that a solution to (60) and (61), together with the free-entry condition (7), exists. This solution will be an equilibrium provided only that the stochastic process for entry of new production firms that is implied takes only nonnegative values. This can be guaranteed provided the stochastic process for aggregate productivity is not
and each \( k \).

Using the matching function, I can calculate the marginal recruiting cost \( k_t^i = \gamma / m(\theta_i) \). For each \( i \) and each \( k \in \{1, 2, \ldots, m_p\} \) and for each \( j \) and each \( l \in \{1, 2, \ldots, m_s\} \), substitute \( n = n_{kl}^* \) and \( n = \bar{n}_{kl} \) into equations (60) and (61). This generates a system of \( 2m_p^2m_s^2 + 2m_pm_s \) linear equations in the \( 2m_p^2m_s^2 + 2m_pm_s \) variables \( \left\{ J'_{ij}(n_{kl}^*) \right\}_{i,j,k,l;}, \left\{ J'_{ij}(\bar{n}_{kl}) \right\}_{i,j,k,l;}, \left\{ n_{kl}^{\alpha-1} \right\}_{k,l;}, \text{and} \left\{ \bar{n}_{kl}^{\alpha-1} \right\}_{k,l;}. \) (There are \( m_p^2m_s^2 \) \( (i, j, k, l) \)-tuples, each of which generates precisely two equations, one using \( n_{kl}^* \) and one using \( \bar{n}_{kl} \), except that an additional two equations is generated for tuples of the form \( (i, j, i, j) \) for which two cases in (61) both provide valid equations.) However, in order to generate the correct equations, it is necessary to know for each \((i, j, k, l)\) whether \( n_{kl}^* < \bar{n}_{ij} \) (in which case a firm which begins a period with \( n_{kl}^* \) workers in productivity state \((p_i, \pi_j)\) immediately hires a positive measure of workers and the first part of (61) applies), whether \( n_{kl}^* \in [\bar{n}_{ij}, \bar{n}_{ij}] \) (in which case neither hiring nor firing occurs and the second case of (61) applies), or whether \( n_{kl}^* > \bar{n}_{ij} \) (in which case the firm fires a positive measure of workers and the third case of (61) applies). That is, the form of the equations is a function of the rank order of the terms of the tuple \( L \equiv (n_{11}^*, n_{12}^*, \ldots, n_{1m_s}^*, n_{21}^*, \ldots, n_{2m_s}^*, \ldots, n_{m_p m_s}^*, \bar{n}_{11}, \bar{n}_{12}, \ldots, \bar{n}_{1m_s}, \bar{n}_{21}, \ldots, \bar{n}_{2m_s}, \ldots, \bar{n}_{m_p m_s}). \) The obvious generalization of the solution algorithm from Section 2.8 can then be used. More precisely, this algorithm is as follows.

1. Guess values for market tightness \( \theta_i \) for each \( i = 1, 2, \ldots, m_p; \) calculate \( \{k_t^i\}_i \).
2. Guess the rank order of the elements of \( L \).
3. Solve the \( 2m_p^2m_s^2 + 2m_pm_s \) equations given by (61) for \( \left\{ J'_{ij}(n_{kl}^*) \right\}_{i,j,k,l;}, \left\{ J'_{ij}(\bar{n}_{kl}) \right\}_{i,j,k,l;}, \left\{ n_{kl}^{\alpha-1} \right\}_{k,l;}, \text{and} \left\{ \bar{n}_{kl}^{\alpha-1} \right\}_{k,l;}. \)
4. Verify whether the order of the solution for \( \left\{ n_{ij}^* \right\}_{ij} \cup \left\{ \bar{n}_{ij} \right\}_{ij} \) is consistent with the guessed order from step 2. If not, guess a new order and return to step 3.
5. Solve the \( 2m_p^2m_s^2 \) equations given by (60) for \( \left\{ J_{ij}(n_{kl}^*) \right\}_{i,j,k,l} \) and \( \left\{ J_{ij} (\bar{n}_{kl}) \right\}_{i,j,k,l}. \)
6. Verify whether the free entry condition holds with equality for every \( i \). If not, guess a new set of market tightnesses \( \{\theta_i\}_i \) and return to step 1.

Again, the algorithm converges rapidly to the correct ordering for \( L \), and the resulting reduced-form equations for \( \{\theta_i\}_i \) can readily be solved by standard nonlinear equation-solving techniques.
4 Quantitative Analysis

Having demonstrated how to solve the model outlined in Section 2 and Section 3, I can now turn to a quantitative analysis of its properties.\textsuperscript{10} Since we observe fluctuations in unemployment and in the vacancy-unemployment ratio, I restrict my discussion of the quantitative properties of the model to an analysis of the model with linear preferences and goods-intensive recruiting outlined in Section 3. I first describe the calibration strategy, and show that the model is able to match the cross-sectional patterns of employment and employment growth. I then examine the cyclical properties of the model, and compare them to those of the benchmark MP model.

4.1 Calibration Strategy

The calibration strategy for the cross-section of the model is similar to that used in Elsby and Michaels (2010) (hereafter EM) to facilitate comparability between my results and theirs. I calibrate parameters largely to match moments pertaining to the steady state, and use the cyclical behavior of the model as a test of its fit. I set the model period to be monthly, and the discount factor to $0.96^{1/12} = 0.9966$. I normalize the units of output by setting the consumption equivalent of the disutility of labor in the production sector, $\phi^h$, to 1. Several standard calibration targets are standard in the labor search and matching literature. I assume that the matching function takes the Cobb-Douglas form $m(\theta) = Z\theta^{-\zeta}$, and choose $Z$ and $\zeta$ so that the worker’s probability of finding a job within any given month, $f(\theta) = \theta m(\theta)$, takes the value 0.45 when the vacancy-unemployment ratio $\theta$ equals 0.72.\textsuperscript{11} I set $\zeta = 0.6$, following Petrongolo and Pissarides (2001) and EM, which requires setting $Z = 0.515$ to be consistent with the desired matching rate for workers. I also choose the exogenous destruction rate so as to match a total monthly separation rate (due to both the endogenous firing by firms receiving adverse productivity shocks and the exogenous job destruction at closing firms) of 3.12\%, consistent with Shimer (2005) as well as EM.

I set the cost of vacancy posting, following Silva and Toledo (2009) and EM, so that the cost of a recruitment opportunity, $\gamma / m(\theta)$, is 42\% of the average monthly wage. I then choose the bargaining power of workers to be 0.5. The bargaining power of workers intuitively maps monotonically to the replacement ratio (that is, the ratio of the disutility of labor, $\phi^h$ – or equivalently, the value of leisure – to the average wage of employed workers). To see this, imagine for a moment that all workers earned the same wage $w$ and that the separation rate $s$ was constant across workers. In steady state in such an environment, the two Bellman equations respectively for a worker unemployed at the beginning of the labor market phase in a particular period and for an employed worker would

\textsuperscript{10}It should be emphasized that the current calibration is preliminary, and subject to modification in later versions of this paper.

\textsuperscript{11}This value follows Pissarides (2009) and EM, but since I do not use data on vacancies as a calibration target, it is simply a normalization. To see this, observe that for $x > 0$, multiplying the scale parameter in the matching function, $Z$, by $x^{x-1}$ and dividing the vacancy-posting cost parameter $\gamma$ by $x$ leads to an equilibrium which differs only in that the equilibrium value of $\theta$ is multiplied by $x$ and the equilibrium value of $m(\theta)$ is divided by $x$. This leaves the matching rate for workers, $f(\theta) = \theta m(\theta)$, the cost of hiring for firms $\frac{\gamma}{m(\theta)}$, unchanged.
take the form

\[ V^u = f(\theta)V^e + (1 - f(\theta))\beta V^u \]  \hspace{1cm} (62)

\[ V^e = w - \phi^h + \beta s V^u + \beta (1 - s) V^e. \]  \hspace{1cm} (63)

In addition, according to the steady-state version of (36) modified for the linear utility function used in the model of Section 3., we know that under the bargaining structure in this paper,

\[ (1 - \beta)V^u = \frac{\eta}{1 - \eta} \cdot \frac{\theta}{1 - f(\theta)} \cdot \frac{\gamma}{m(\theta)}. \]  \hspace{1cm} (64)

Equations (62), (63), and (64) can be rearranged to yield that

\[ \frac{\phi^h}{w} = 1 - \frac{\eta}{1 - \eta} \cdot \frac{1 - \beta(1 - f(\theta))(1 - s) \gamma/m(\theta)}{w}. \]  \hspace{1cm} (65)

Since \( f(\theta), s, \) and \( \gamma/m(\theta)/w \) are all calibration targets, it follows that \( \eta \) is monotonically related to the replacement ratio, and in this simplified model, \( \eta = 0.5 \) corresponds to a replacement ratio of 0.640. (In the full calibrated model, the presence of wage dispersion makes this calculation not exactly correct; the replacement rate there turns out to be 0.629.)

Following EM, I assume that the process for idiosyncratic productivity \( \pi \) takes the form \( \pi = \pi^f \pi^x \), where \( \pi^f \) is a fixed component drawn at the time of the firm’s birth and which does not change, and \( \pi^x \) is a mean-reverting transitory component. In this representation, the process for \( \pi^x \) governs the distribution of employment growth; for this \( \pi^f \) is irrelevant. (To see that \( \pi^f \) is irrelevant for the distribution of employment growth, suppose that two firms are born at the same time, one with permanent shock \( x > 1 \) times the other, and both draw the same sequence of realizations of \( \pi^x \). Let the optimal sequence of employment choices by the firm with the lower value of \( \pi^f \) by \( \hat{n}_t \). Then it is easy to verify that second firm always optimally sets its employment equal to \( x^{1/\alpha}\hat{n}_t \). It follows that changes in log employment are identical across the two firms. Note also that both firms will pay the same wages, according to (32).) I follow EM again by assuming that the stochastic process for \( \pi^x \) is such that with probability \( \lambda \), \( \pi^x_{t+1} = \pi^x_t \), while with probability \( 1 - \lambda \), \( \pi^x_{t+1} \) is drawn independently of \( \pi^x_t \) (as well as of the aggregate state and of the productivity of other firms) from a fixed distribution which is Pareto\((m_x, k_x)\), that is, with density \( G(\pi) = k_x m_x^{k_x} \pi^{k_x+1} \) on \([m_x, \infty)\) and zero elsewhere. I choose \( m_x \) to normalize the expected value of \( \pi^x \) to one, and \( k_x \) and \( \lambda \) to match the moments reported by EM for the Longitudinal Business Database for 1992-2005, namely that each year 37.2% of firms do not change their employment, while the standard deviation of (log) employment growth is 0.416. I discretize the probability distribution for \( \pi^x \), using in the current version of the paper a 75-point grid.

As already observed, the distribution of \( \pi^f \) is irrelevant for the cross-sectional distribution of log employment growth, as well as for the cyclical properties of aggregate variables, so it can be chosen so as to match properties of the cross-sectional distribution of employment. As in EM, I assume that the distribution of \( \pi^f \) is Pareto\((m_f, k_f)\). I choose \( m_f \) to match a minimum firm size
of 1 worker, and set $k_f = (1 - \alpha)$ so that the upper tail of the employment distribution satisfies a power law with exponent $k_f / (1 - \alpha) = 1$, consistent with Zipf’s law (Axtell, 2001). Note that this implies the moments of firm-level employment are not defined.

The only remaining parameter value relevant for the cross-section of the model is $\alpha$, the returns-to-scale parameter in the production function. Various possible parameters might be suggested depending on the source of decreasing returns in the model. Thinking of capital as fixed (over the entire life of the firm) suggests choosing $\alpha$ around 0.67, the labor share. Thinking of decreasing returns as arising from Lucas (1978) span-of-control suggests a much higher value for $\alpha$. I take an intermediate value of $\alpha = 0.8$; future versions of the paper will examine the robustness of the current results to this choice.

I choose the entry cost $K$ so that entering firms make zero profits in the steady state allocation just described.

Moving to the stochastic model, I need to parameterize the stochastic process for aggregate productivity, $p_t$. I use the method suggested by Stokey (2009, Section 2.8) to approximate an Ornstein-Uhlenbeck process. Specifically, I assume that $\log p_t$ takes $2M + 1$ possible values, on the equally-spaced grid $\{-Md, -(M - 1)d, \ldots, Md\}$ for $d > 0$. I assume that there is $\mu \in (0, (2M)^{-1})$ so that if $\log p_t = \kappa d$ for $\kappa \in \{-M, -M - 1, \ldots, M\}$, then $\log p_{t+1}$ takes the value $\log p_t$ with probability $1 - 2M\mu$, $\log p_t - d$ with probability $(M + \kappa)\mu$, and $\log p_t + d$ with probability $(M - \kappa)\mu$. I choose $M = 2$ for the preliminary calibration reported here, and choose $d$ and $\mu$ so that model-generated GDP (aggregated to quarterly frequency and considered as a log deviation from an HP trend with smoothing parameter 1600) match the empirical volatility and autocorrelation of US GDP per capita from 1947Q1 through 2010Q4. Finally, I assume that the cost of entry varies with aggregate productivity (otherwise, entry is too volatile in the model). I obtain data from the Business Employment Dynamics database from the Bureau of Labor Statistics on establishment openings between 1992Q3 and 2010Q2, and regress the (seasonally adjusted and detrended) data on the detrended GDP per capita data already mentioned. The point estimate for the elasticity of establishment openings with respect to GDP per capita is 1.03, and is statistically significant.\footnote{The unavailability of data on establishment openings predating 1992Q3 at higher than annual frequency necessitates the use of GDP per capita as a cyclical measure, rather than the more common output per worker. It is well known that the cyclicity of output per worker sharply decreased around 1990, so that it is much less highly correlated with GDP per capita and much less highly negatively correlated with unemployment. Similarly, the point estimate for the elasticity of establishment openings with respect to detrended output per worker is negative and insignificant. It is beyond the scope of the current paper to speculate on why the cyclicity of labor productivity appears to have changed.} This requires setting $\xi = 0.108$ in order for the model to replicate it.

The resulting parameter values are reported in Table 1.

### 4.2 Results

#### 4.2.1 Cross-Sectional Properties

Figure 1 shows a representative example of the hiring and firing behavior of a single firm. The
Table 1: Parameterization

<table>
<thead>
<tr>
<th>Variable</th>
<th>Value</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>0.997</td>
<td>Annual discount factor 0.96</td>
</tr>
<tr>
<td>$\phi^y$</td>
<td>1</td>
<td>Normalization</td>
</tr>
<tr>
<td>$\zeta$</td>
<td>0.6</td>
<td>Petrongolo and Pissarides (2001)</td>
</tr>
<tr>
<td>$Z$</td>
<td>0.515</td>
<td>Job-finding probability 0.45 at $\theta = 0.72$</td>
</tr>
<tr>
<td>$\delta$</td>
<td>0.0106</td>
<td>Separation probability 3.12%</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>0.418</td>
<td>Recruiting cost 42% of wage</td>
</tr>
<tr>
<td>$\eta$</td>
<td>0.5</td>
<td>Replacement ratio 0.64</td>
</tr>
<tr>
<td>$m_x$</td>
<td>0.804</td>
<td>Normalization: $E\pi^B = 1$</td>
</tr>
<tr>
<td>$k_x$</td>
<td>5.101</td>
<td>Std. dev. employment changes 0.416</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>0.339</td>
<td>Annual prob. no change in employment 37%</td>
</tr>
<tr>
<td>$m_f$</td>
<td>2.219</td>
<td>Minimum establishment size 1 worker</td>
</tr>
<tr>
<td>$k_f$</td>
<td>5</td>
<td>Zipf’s law</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.8</td>
<td>See text</td>
</tr>
<tr>
<td>$K$</td>
<td>0.821</td>
<td>See text</td>
</tr>
<tr>
<td>$M$</td>
<td>2</td>
<td>See text</td>
</tr>
<tr>
<td>$d$</td>
<td>0.0169</td>
<td>Variance of GDP per capita</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>0.0279</td>
<td>Autocorrelation of GDP per capita</td>
</tr>
<tr>
<td>$\xi$</td>
<td>0.108</td>
<td>Cyclicity of establishment openings</td>
</tr>
</tbody>
</table>

The blue line indicates the time path of the firm’s number of employees, $n(t)$. The green line indicates the hiring target, $n_j^*$ corresponding to the current idiosyncratic productivity state $j$ at time $t$. The red line indicates the time path of the firing target $\bar{n}_j$. As can be seen, whenever the hiring target increases to a level that exceeds the size of the firm in the preceding period, hiring occurs to the new value of the hiring target. Similarly, when the firing target falls below the current size of the firm, firing occurs so as to keep the current size no greater than that target. Otherwise, if current employment lies between the two targets, the firm does not change its employment level.

Figure 2 shows the cross-sectional ergodic distribution of firm size in the steady state conditional on the value of the permanent component of the idiosyncratic shock $\pi^f$. The distribution is plotted for the smallest value of $\pi^f$, so that the minimum realized employment level is 1. Note that substantial dispersion arises from the highly dispersed idiosyncratic shock. Figure 3 shows a histogram for employment growth rates $g_t$, defined as in Davis and Haltiwanger (1990, 1992) by

$$g_t = \frac{\log(n_t) - \log(n_{t-1})}{\frac{1}{2}(\log(n_t) + \log(n_{t-1}))}.$$

The Figure shows the substantial measure of firms which do not change employment, together with

\[13\] Recall that two firms with different values of $\pi^f$ have employment paths that differ only by a constant log increment, so that the empirical counterpart to this measure in a panel data set with a long enough time series dimension is the distribution of employment. The ergodic theorem does not precisely apply to the distribution over time of the employment of an individual firm since it is only born once and newborn firms tend to be smaller; however, taking averages over many firms with the same average lifetime employment would generate the distribution in Figure 2 in an infinite-length panel.
the large dispersion in employment growth, both of which were targeted by the calibration.

Figure 4 shows the unconditional distribution of firm size. Because this distribution obeys Zipf’s law in the tail, Figure 4 shows plots the log of firm size against the logarithm of one minus the cumulative distribution function. In the left part of the distribution, the distribution does not obey a power law because very small firms must have both a low value for the transitory component of idiosyncratic productivity as well as a low value for the permanent component, so that the shape of the left tail of the distribution conditional on the permanent component (Figure 2) affects the shape of the unconditional distribution more strongly.

The model is consistent with a positive cross-sectional correlation between firm size and wages, as Oi and Idson (1999) report for US data. The correlation arises entirely from a positive relationship between wages and employment conditional on the permanent component of the idiosyncratic productivity shock; as already mentioned, a firm with higher permanent productivity hires more workers than, but pays the same wages as, a firm with lower permanent productivity. After controlling for the permanent component of productivity, I regress log wages on log employment and a constant; the resulting regression relationship is

\[ \log(\text{wage})_{it} = 0.676 + 0.073 \log(\text{employment})_{it} + \varepsilon_{it} \]

with \( R^2 = 0.61 \) and \( \text{Var}(\varepsilon_{it}) = 0.0041 \).\(^{14}\)

The model is also consistent with the empirical positive relationship between wages and firm growth rates. This positive correlation between wages and firm growth might seem to be counterintuitive, since the wage-setting assumption implies that conditional on productivity, firms with higher employment pay lower wages because the marginal product of the last worker is lower. However, this observation ignores the endogeneity of employment. If there were no labor market friction, all firms would hire and fire up to the point that the bargained wage was equated across firms (and in fact equal to the worker’s outside option, as in Stole and Zwiebel (1996a,b)). In the presence of costly recruiting, firms that receive a positive productivity shock do not hire enough workers to drive the wage so far down, so that such firms pay higher wages, at least conditional on employment.\(^{15}\)

Finally, the model is also consistent with a further stylized feature of the employment growth distribution. Haltiwanger et al. (2010) report that young businesses and startups create a disproportionate fraction of jobs in the US economy. The model replicates this stylized fact successfully: young firms hire many employees at entry as they grow to their target size. Moreover, a firm that initially draws a low value of the transitory component of idiosyncratic productivity expects to increase in size in the future as it draws higher values. As the firm ages, its employment level is

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\(^{14}\)I do not report standard errors because there is no sampling error: the ergodic distribution is calculated exactly. The positive relationship is robust to the inclusion of higher moments of log employment.

\(^{15}\)Because I assume that entrant firms draw from the same transitory productivity distribution as incumbents, this mechanism renders the model inconsistent with the fact that younger firms on average pay lower wages (Davis and Haltiwanger, 1991). This could easily be rectified by assuming that the productivity distribution for entrants is dominated (in the sense of first-order stochastic dominance) by that for incumbents.
Figure 1: Firm size dynamics

Figure 2: CDF of firm size conditional on permanent component of idiosyncratic productivity
Figure 3: Histogram of firm employment growth

Figure 4: Unconditional CDF of firm size
more likely to lie in the inaction region, and its growth rate declines. In addition, because firm size is largely driven by the permanent component of idiosyncratic productivity, the model is also consistent with the finding that there is no systematic relationship between firm sizes and growth rates, also reported by Haltiwanger and coauthors. (A future version of this paper will evaluate this quantitatively.)

4.2.2 Business Cycle Properties

I now turn to describing the business-cycle features of the model. The first feature to note is that, as established theoretically, the vacancy-unemployment ratio is a jump variable whose value is affected only by contemporaneous productivity. Figure 5 shows a sample path for the dynamics of this variable.

Figure 6 shows the dynamics for a sample path for three other key endogenous variables, specifically, GDP per capita, unemployment, and entry, and Table 2 reports the standard deviations, autocorrelations, and correlation matrix for these three variables together with aggregate productivity, vacancies, and the vacancy-unemployment ratio. To generate the figure and the table, I aggregate the monthly model-simulated data to quarterly frequency, then calculate log deviations from each variable’s mean. I do not remove any other trend, so as to avoid introducing spurious autocorrelation arising from the introduction of a spurious time trend in the HP filtering process.

From Figure 6 the most notable feature is the spikes in firm entry at the arrival of an aggregate
productivity shock. This behavior arises since vacancies have to jump on the arrival of an aggregate shock in order to make the vacancy-unemployment ratio jump to its new value while unemployment does not immediately change. The fraction of this increase in vacancies that arises on the extensive margin (vacancies posted by new entrant firms) as opposed to the intensive margin (vacancies posted by incumbents) is governed by the parameter $\xi$, the elasticity of the entry cost with respect to aggregate productivity. Raising $\xi$ would reduce the magnitude of the spikes in entry, and would reduce the volatility of entry (in the model, the standard deviation of log entry is 0.074, while in the Business Employment Dynamics database for 1992Q3-2010Q2, the analogous value is 0.034). However, this would come at the cost of worsening the model’s fit to the target used to calibrate $\xi$, namely the elasticity of entry with respect to changes in GDP per capita. Improving the empirical performance of the model along this dimension could be accomplished by introducing a ‘time-to-build’ feature along the lines of Fujita and Ramey (2007); this might also help with making the vacancy-unemployment ratio respond with a lag to productivity shocks, as in their paper, although the implication is not clear since in my model incumbent firms can also respond to a shock by posting more vacancies. A further examination of this modification is beyond the scope of the current paper.

Beyond the behavior of entry, the results reported in Table 2 emphasize that the dynamic behavior of the model is extremely close to that of the benchmark MP model, as reported in Shimer (2005) and Hagedorn and Manovskii (2008). Table 2 is rather similar to Shimer’s Table 3. The moment most readers will probably focus on most is the ratio of the standard deviation of the v-u ratio to that of productivity. This ratio is around 1.75 in Shimer’s model, and around 2.58 in mine. However, recall that I calibrated the replacement ratio to around 0.63, higher than Shimer’s value of 0.4. Using the back-of-the-envelope elasticity formulae provided on page 36 of Shimer’s paper, one can see that this change alone would tend to increase the elasticity of unemployment with respect to productivity shocks in the MP model by a factor of around $(1 - 0.4)/(1 - 0.63) \approx 1.62$. Observe that $1.62 \times 1.75 \approx 2.84 > 2.58$, so that when this difference is controlled for, my model actually generates less volatility in the v-u ratio than the benchmark MP model! The other obvious difference between Table 2 and Shimer’s Table 3 is that the Beveridge curve is less pronounced: vacancies are less (negatively) correlated with unemployment. This appears to result from the spikes in vacancies discussed above. Otherwise the performance of the model is not noticeably different than that of the benchmark MP model for those variables described by both models. In particular, because of the single driving force in the model and the absence of a propagation mechanism, all the reported variables, with the exception of entry, are highly correlated with aggregate productivity.

5 Discussion

As noted in the Introduction, the question of whether a firm-size distribution that responds sluggishly to shocks provides a channel for propagating or amplifying aggregate shocks has not been unanimously resolved in the literature to date. The current paper therefore casts in starker relief
Figure 6: Dynamics of GDP per capita and unemployment

<table>
<thead>
<tr>
<th></th>
<th>u</th>
<th>v</th>
<th>v/u</th>
<th>e</th>
<th>p</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard deviation</td>
<td>0.030</td>
<td>0.019</td>
<td>0.044</td>
<td>0.074</td>
<td>0.017</td>
<td>0.020</td>
</tr>
<tr>
<td>Quarterly autocorrelation</td>
<td>0.942</td>
<td>0.609</td>
<td>0.898</td>
<td>0.582</td>
<td>0.898</td>
<td>0.907</td>
</tr>
<tr>
<td>Correlation matrix</td>
<td>u</td>
<td>v</td>
<td>v/u</td>
<td>e</td>
<td>p</td>
<td>y</td>
</tr>
<tr>
<td>u</td>
<td>1</td>
<td>-0.614</td>
<td>-0.938</td>
<td>0.029</td>
<td>-0.937</td>
<td>-0.947</td>
</tr>
<tr>
<td>v</td>
<td>1</td>
<td>0.850</td>
<td>0.735</td>
<td>0.850</td>
<td>0.834</td>
<td></td>
</tr>
<tr>
<td>e</td>
<td>1</td>
<td>0.304</td>
<td>1.000</td>
<td>0.999</td>
<td></td>
<td></td>
</tr>
<tr>
<td>p</td>
<td>1</td>
<td>0.304</td>
<td>0.278</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>y</td>
<td>1</td>
<td>1.000</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2: Parameterization
what features of the models of Acemoglu and Hawkins (2010), Elsby and Michaels (2010), and Kaas and Kircher (2011) are important for generating propagation and amplification. (I hereafter refer to these papers as AH, EM, and KK respectively.)

The first important difference among the four papers is the nature of returns to scale in the recruiting technology. AH and KK both assume that a firm that wishes to recruit progressively more workers in a given period faces a convex cost schedule for doing so, while the current paper, along with EM, assumes a constant returns vacancy-posting technology. The propagation found by AH and by KK, but not by the current paper, appears to arise from this feature of the model. In response to an aggregate shock, incumbent firms may change their size target. However, arriving at the new size target instantaneously is very costly, so that at the level of the individual firm, there are slow adjustment dynamics. This maps to sluggish aggregate dynamics for the vacancy-unemployment ratio. By contrast, in the model of the current paper, there is no such sluggishness at the level of the individual firm. Moreover, the changing composition of hiring at the intensive and extensive margin over the cycle provides a margin by which the desire of incumbent firms to hire en masse in response to a shock can be accommodated. The resulting aggregate dynamics therefore display no persistence in the vacancy-unemployment ratio.

Propagation arises in EM for a different reason. There, the vacancy-posting technology does exhibit constant returns, but the crucial difference with the current paper is that the number of firms is fixed. Consider the response to a negative productivity shock. In the model of the current paper, entry falls sharply on the arrival of the shock, with vacancy-posting by incumbents rising to take up some of the slack. Wages fall as incumbent firms increase in size, so their marginal product decreases. Once wages have fallen, entry by new firms returns nearly to its pre-shock level. In the absence of an entry margin, the response is different, however. Suppose that the vacancy-unemployment ratio were to fall instantly to the post-shock quasi-steady state level. Then, because unemployment rises during the transition to the new quasi-steady state, vacancies would also have to fall on impact and then rise thereafter. However, on impact of the shock, wages fall as the outside option of being unemployed falls with the job-finding rate and with the fall in aggregate productivity, so that incumbent firms instead wish to increase vacancy posting at the arrival of the shock. This inconsistency means that the fall in vacancy-posting on impact must be more gradual, and this leads to persistence.

Another difference between EM and the three other papers mentioned is that they find substantial amplification of productivity shocks. However, much of the difference in this respect arises because in their calibration, a typical firm would, if it hired a worker, generate a fairly small surplus of employment. (The replacement ratio is around 0.85 in their calibration.) Additionally, the absence of an extensive margin of adjustment again works to generate increased amplification, due to the role of the wage bargaining assumption. Following a positive productivity shock, not only do firms wish to hire more workers simply because of the direct increase in profitability due to the shock, but also because the rise in wages for all workers is offset as larger firms pay lower wages conditional on productivity. As the average size of firms increases following the shock, the rise
in wages due to the productivity shock is partially offset. This dampening effect on the elasticity of wages due to the productivity shock acts as would an exogenously sticky wage, increasing the response of employment to the shock. By contrast, in the current model, the adjustment to a positive productivity shock occurs largely on the extensive margin: more establishments enter, and the size of incumbent firms barely changes. Thus, this second feedback channel is absent. (Quantitatively, however, the more important difference between the two papers lies in the calibration of the replacement ratio.)

6 Conclusion

The key contributions of this paper are twofold. The first is theoretical. I solve, nearly in closed form, a large-firm bargaining model in the presence of a rich structure of cross-sectional heterogeneity. The second is more quantitative. I show that in this class of models, the ability to match cross-sectional features of the employment and employment growth distribution has no implications for the response of the model to aggregate shocks. Indeed, if preferences take a balanced-growth form and recruiting is time-intensive, then no matter what the cross-sectional features of the model in steady state, there will be no fluctuations in unemployment in response to an aggregate productivity shock! In the more familiar case with linear preferences and goods-intensive recruiting, the responses of the key aggregate variables described by the MP model are not significantly different in my model. The presence of large firms and of a rich structure of cross-sectional heterogeneity need not amplify or propagate business-cycle shocks. To propagate shocks, it appears that either convex costs of recruiting or shutting down the extensive margin of adjustment to shocks is required. To amplify shocks seems to require that the surplus from employment be small, something which is unsurprising since the work of Shimer (2005) and Hagedorn and Manovskii (2008).

A significant success not just of the current paper, but also of the quantitatively-oriented contributions of Elsby and Michaels (2010) and Kaas and Kircher (2011) is to match the empirical cross-sectional patterns of employment and employment growth well. However, a concern about all three papers is that the calibrational strategy that underlies this success is not well-disciplined: in the calibration, the productivity distribution is essentially a residual. A natural direction for future research in theoretical modeling is to determine to what extent the results of the three papers are dependent on the adjustment cost structure (what would happen in the presence of a fixed cost of adjusting the labor force?) A natural direction for empirical research is to attempt to ground the productivity distribution more directly in the data. Both challenges are beyond the scope of the current paper.

A Omitted Proofs

This Appendix contains statements of some definitions and results that are not formalized in the text, together with all technical proofs. Some proofs are not yet complete in this preliminary version
of the paper.

Proof of Lemma 2. Define

\[ T(n) = \frac{1}{\eta} \int_0^n \nu^{\frac{1-n}{n}} y'(\nu) \, d\nu = \frac{\int_0^n \nu^{\frac{1-n}{n}} y'(\nu) \, d\nu}{\int_0^n \nu^{\frac{1-n}{n}} \, d\nu}. \] (66)

From the definition of \( \tilde{y}(\cdot) \) in (34), it is apparent that \( \tilde{y}(n) = y(n) - nT(n) \). Differentiating establishes also that \( \tilde{y}'(n) = (1 - \eta)T(n) \) and \( \tilde{y}''(n) = \frac{1-n}{n} [y'(n) - T(n)] \). Thus, to establish the desired claims about \( \tilde{y}(\cdot) \), it is sufficient to show that \( y'(n) < T(n) < \frac{y(n)}{n} \) for all \( n \). This establishes directly that \( \tilde{y}(n) \) is strictly positive and that \( \tilde{y}''(\cdot) \) is strictly negative. That \( \tilde{y}'(n) > 0 \) follows for \( \text{fortiori} \) because \( y'(n) > 0 \). The two Inada conditions also follow: as \( n \to 0^+ \), \( T(n) > y'(n) \to +\infty \), and as \( n \to +\infty \), \( 0 < T(n) < \frac{y(n)}{n} \to 0 \).

To see that \( T(n) > y'(n) \), note that the last expression in (66) shows that \( T(n) \) is a weighted average of inframarginal products \( y'(\nu) \) for \( \nu \in (0, n) \). Since \( y(\cdot) \) is strictly concave, \( y'(\nu) \) strictly exceeds \( y'(n) \) whenever \( 0 \leq \nu < n \) since \( y'(\cdot) \) is a strictly decreasing function; that is, \( T(n) > y'(n) \) for all \( n \).

To see that \( T(n) < y(n)/n \), write

\[ y(n)/n = \frac{\int_0^n y'(\nu) \, d\nu}{\int_0^n \nu \, d\nu}. \] (67)

Comparing the last expression in (66) with (67), it is apparent that both \( y(n)/n \) and \( T(n) \) are weighted averages of \( y'(\nu) \) for \( \nu \in (0, n) \). However, the weighting function \( \nu \to \nu^{(1-\eta)/\eta} \) in (66) is strictly increasing in \( \nu \), while that in (67) is a constant function. Because the integrand \( y'(\nu) \) is strictly decreasing in \( \nu \), the weighting function that puts relatively higher weights on lower values of \( \nu \), that is, the constant function, yields a higher weighted average. This completes the proof. \( \square \)

Proof of Proposition 1. I show that amongst the allocations which satisfy the hypotheses of Proposition 1 (that is, stationary allocations), I can find one which is an equilibrium. The proof consists of several parts. First, I normalize the problem of a production firm in a stationary allocation to make it independent of aggregate productivity. Second, I construct values of the key endogenous variable \( \bar{c} \) and \( \bar{\theta} \) that are consistent with equilibrium in the labor market, equilibrium in the goods market, and free entry for production firms. Finally, I show that there are values for the remaining endogenous variables consistent with equilibrium.

Therefore, consider an arbitrary stationary allocation. Observe first that \( x(s^t) \) is constant since \( \theta(s^t) \equiv \bar{\theta} \), so that from (36) it follows that in a stationary allocation,

\[ x(s^t) \equiv \bar{x} = \frac{\bar{\theta} \eta \gamma \phi^r}{(1 - f(\bar{\theta}))(1 - \eta)} \] (68)

is a function only of \( \bar{\theta} \) and parameters. Note that \( \bar{x} \) is strictly increasing in \( \bar{\theta} \).

Next, define normalized value functions for firms by dividing equations (4) and (5) by current aggregate productivity \( \rho_t \), so that the normalized value of a firm after learning the realizations of productivity but before firing or recruitment is

\[ \hat{K}(n; s^t, \pi^t) = \max_{f, h \geq 0} \left[ -\frac{k^r(s^t)}{\rho_t} h + K(n - f + h; s^t, \pi^t) \right] \] (69)
By assumption of stationarity, \( k^r(s^t) = \bar{k}^r p_t \). Also, from (30),
\[
\frac{w(n; s^t, \pi^t)}{p_t} = (1 - \eta)(\phi^y + \bar{x})\bar{c} + n^{-\frac{1}{\eta}} \int_0^n \nu \pi \pi_j y'(\nu) d\nu. \tag{71}
\]
That is, the ratio of wages to aggregate productivity depends only on current employment and idiosyncratic productivity at the firm. Denote this quantity (the right side of (71)) as \( \bar{w}(n, \pi_t) \) hereafter. Finally, (15), together with stationarity, implies that
\[
q_t(s^{t+1})\frac{p_{t+1}}{p_t} = \beta \frac{P(s^{t+1})c(s^t) p_{t+1}}{P(s^t) c(s^{t+1}) p_t} = \beta \frac{P(s^{t+1})}{P(s^t)}. \tag{72}
\]
Substituting all these observations into (69) and (70) establishes that
\[
\hat{K}(n; s^t, \pi^t) = \max_{f, h \geq 0} \left[ -\bar{k}^r h + K(n - f + h; s^t, \pi^t) \right] \tag{73}
\]
and
\[
K(n; s^t, \pi^t) = \pi_t y(n) - n\bar{w}(n, \pi_t) + \beta(1 - \delta) \sum_{s^{t+1} | s^t} \frac{P(s^{t+1})}{P(s^t)} \sum_{\pi^{t+1}} \mu(\pi_t, \pi_{t+1}) \hat{K}(n; s^{t+1}, \pi^{t+1}). \tag{74}
\]
Now, suppose the solution for \( \hat{K} \) and for \( K \) is in fact independent of the history \( s^t \). In this case the only terms involving \( s^t \) are the weights \( \frac{P(s^{t+1})}{P(s^t)} \) on the right side of (73); however, these weights and the entire summation are superfluous since the summand is constant with respect to \( s^{t+1} \). That is, \( \hat{K} \) and \( K \) are a history-independent solution to (72) and (73) if and only if they satisfy
\[
\hat{K}(n, \pi_t) = \max_{f, h \geq 0} \left[ -\bar{k}^r h + K(n - f + h, \pi_t) \right] \tag{75}
\]
and
\[
K(n, \pi_t) = \pi_t y(n) - n\bar{w}(n, \pi_t) + \beta(1 - \delta) \sum_{\pi_{t+1}} \mu(\pi_t, \pi_{t+1}) \hat{K}(n, \pi_{t+1}). \tag{76}
\]
(Here I abuse notation and drop redundant arguments of the two value functions.) (74) and (75) can be combined as the single equation
\[
\hat{K}(n, \pi) = \max_{n'} \left[ -\bar{k}^r \max(n' - n, 0) + \pi \bar{y}(n') - n'(1 - \eta)(\phi^y + \bar{x})\bar{c} + \beta(1 - \delta) \sum_{\pi_{t+1}} \mu(\pi_t, \pi_{t+1}) \hat{K}(n', \pi_{t+1}) \right] \tag{77}
\]
where \( \bar{y}(n) \) was defined in (34). The discussion there established the strict concavity of \( \bar{y}(\cdot) \). Also, Blackwell’s sufficient conditions apply, so that there is a unique solution to (77) (and therefore to (74) and (75)). This solution then solves (72) and (73), and therefore generates a solution to (4) and (5). (It is unique under weak assumptions on the stationarity of the aggregate productivity process, but formalizing this is unnecessary for the current argument.)

Because \( \bar{y}(\cdot) \) is a neoclassical production function, (77) can be viewed as characterizing the
A strictly decreasing continuous function with \( \lim_{\bar{\psi} \to \psi} \psi \) vanish. Therefore if I write free entry condition, we must have \( \bar{\psi} \) for the firm, and then only if \( \bar{\psi} \) is a finite upper bound \( \bar{\psi} \) since \( \bar{\psi} \psi \) and labor market clearing in an economy where consumption is \( \bar{\psi} \theta \). A similar argument establishes the other claim.

During the production phase of the economy always at the static optimum, which satisfies of the firm, so that there are no costs of adjusting the firm’s labor force, and it sets employment continuously, in the sense that there is no \( \bar{\psi} f \) the job-finding probability of workers, \( \bar{\psi} \bar{r} \) that is consistent with labor market clearing in this environment.

In a stationary allocation, (8) and (13) imply that \( \bar{k} f = \kappa \phi^f \bar{c} \) and (9) and (14) imply that \( \bar{k} r = \gamma \phi^r \bar{c} / m(\bar{\theta}) \).

I now show that there are values of \( \bar{c} \) and \( \bar{\theta} \) consistent with market clearing in the labor and goods markets and with free entry.

The problem of a neoclassical firm solving (76) is well-understood. The value of such a firm is continuously decreasing in \( \bar{\theta} \) and in \( \bar{c} \). (To prove the first of these claims more formally, parameterize the solution for different candidate values of \( \bar{\theta} \) by \( \bar{K}(n, \pi; \bar{\theta}) \), and note that if \( \bar{\theta}_1 > \bar{\theta}_2 \), then \( \bar{k} r (\bar{\theta}_1) > \bar{k} r (\bar{\theta}_2) \) (it is more expensive to recruit when the recruiter to unemployment ratio is high) and \( \bar{x}(\bar{\theta}_1) > \bar{x}(\bar{\theta}_2) \) (wages are higher). A production firm in an economy with \( \bar{\theta} = \bar{\theta}_1 \) can always follow the policy that is optimal in an economy with \( \bar{\theta} = \bar{\theta}_2 \). This will achieve a higher value since the cost of the recruiting the firm does is reduced, as are wages. Further optimizing recruitment strategies can only increase the firm’s value further. Moreover, this increase in the value occurs continuously, in the sense that there is no \( \bar{\theta}_1 \) at which the value of an entrant, which is given by \( \sum_{j=1}^{m_n} F_{\pi}^e(\zeta_j) \bar{K}(0, \zeta_j) = \bar{k} f \) increases discretely. This is because if \( \bar{\theta}_2 \to \bar{\theta}_1 \), a firm in the economy with \( \bar{\theta} = \bar{\theta}_2 \) could also follow the firing and recruitment policies that are optimal in the \( \bar{\theta} = \bar{\theta}_1 \) economy. A similar argument establishes the other claim.

Moreover, the largest value of \( \bar{c} \) that is consistent with the free entry condition is obtained when \( \bar{\theta} \to 0 \), so that \( m(\bar{\theta}) \to +\infty \) and \( \bar{k} r \to 0 \). In this case, recruitment has zero cost from the perspective of the firm, so that there are no costs of adjusting the firm’s labor force, and it sets employment during the production phase of the economy always at the static optimum, which satisfies

\[
\bar{y}'(n) = (1 - \eta) \phi^y \bar{c} \tag{78}
\]

since \( \bar{x} \to 0 \) in this case. Because \( \bar{y}(\cdot) \) is a neoclassical production function by Lemma 2, it follows there is an finite upper bound \( \bar{c}_1 \) which is the largest value consistent with the free entry condition for the firm, and then only if \( \bar{\theta} = 0 \). On the other hand, as \( \bar{c} \to 0^+ \), in order to be consistent with the free entry condition, we must have \( \bar{\theta} \to +\infty \) so that recruiting becomes expensive, since if \( \bar{\theta} \) remains bounded, so does \( \bar{x} \), so that the firm faces no other costs of operation as wages \( (1 - \eta)(\phi^y + \bar{x}) \bar{c} \) vanish. Therefore if I write \( \psi(\bar{c}) \) for the value of \( \bar{\theta} \) consistent with (77), then \( \psi : (0, \bar{c}_1) \to [0, \infty) \) is a strictly decreasing continuous function with \( \lim_{\bar{c}_1 \to 0^+} \psi(\bar{c}) = +\infty \) and \( \psi(\bar{c}_1) = 0 \).

Next, define a function \( \bar{\xi} : (0, \bar{c}_1] \to [0, \infty) \) as the level of output consistent with firm behavior and labor market clearing in an economy where consumption is \( \bar{c} \) and the recruiter-unemployment ratio is \( \psi(\bar{c}) \). To define this more precisely, first denote by \( G(n, \zeta_j) \in [0, 1] \) the probability a firm has employment \( n \) and idiosyncratic shock \( \zeta_j \) during the production phase of a period. Denote by \( g(\bar{c}) \) the measure of active firms (after entry and before exogenous destruction within any period) that is consistent with labor market clearing in this environment. \( g(\bar{c}) \) is uniquely determined by the job-finding probability of workers, \( f(\bar{\theta}) = f(\psi(\bar{c})) \), together with the firing behavior of firms,
determined as part of the solution to the firm’s problem, and the distribution $\bar{G}(\cdot)$.\footnote{More precisely, after the recruitment phase of a period, total employment is}

$$\xi(\bar{c}) = g(\bar{c}) \sum_{j=1}^{m_x} \int_0^n y(n) \, dF(n, \zeta_j). \quad (79)$$

Since all output is consumed (some after being paid as wages directly, some paid via construction or recruitment firms which make zero profits and operate technologies which use labor as the only input), a necessary condition for equilibrium is that $\xi(\bar{c}) = \bar{c}$. It is apparent that $\xi$ is continuous. Therefore it suffices to show that if $\bar{c} \to 0$ then $\xi(\bar{c}) > \bar{c}$ and if $\bar{c} \to \bar{c}_1$ then $\xi(\bar{c}) < \bar{c}$; the result then follows by the intermediate value theorem. But if $\bar{c} \to 0$, then $\psi(\bar{c}) \to +\infty$, so that workers’ job-finding probability $f(\psi(\bar{c}))$ is strictly positive, and therefore so is output. If $\bar{c} \to \bar{c}_1$, then $\psi(\bar{c}) \to 0$, so that workers’ job-finding probability $f(\psi(\bar{c})) \to 0$, so that the fraction of employed workers becomes small; moreover, since in this case firm’s employment policy is given by (78), employment per firm does not vanish, so that output per employed worker remains bounded. Thus output converges to zero as $\bar{c} \to \bar{c}_1$. This completes the proof of the existence of a $(\bar{c}, \bar{\theta})$ pair that is consistent with clearing the labor and goods markets and with firm free entry.

Construction of the rest of the equilibrium is now routine. In the argument above, the number of entering firms and the optimal hiring and firing policies of firms was constructed; the amount of entry and recruitment is then readily constructed, as is unemployment and the firm size distribution. The normalized cost of a factory is $\bar{k}^f = \kappa \phi^f \bar{c}$ and the normalized cost of hiring a worker is $\bar{k}^r = \gamma \phi^r \bar{c} / m(\bar{\theta})$. The normalized wage in the construction sector is $\bar{w}^f = \phi^f \bar{c}$ from (8), and the normalized wage in the recruitment sector is $\bar{w}^r = \phi^r \bar{c}$ from (9). The normalized wage in the production sector, $\bar{w}(n, \pi)$ is given by the right side of (71). By construction, all firms optimize, and households choose optimally their labor supply to the construction and recruitment sectors. The free entry condition holds with equality. Clearance in the goods market implies that the household chooses consumption and savings optimally. That is, the allocation so constructed is an equilibrium. This completes the proof. \hfill $\Box$

Proof of Proposition 2. The argument is analogous to that given in the proof of Proposition 1. The only difficulty is to show that values \{\$\theta_i\}, can be found so that the free-entry condition (7) holds for all aggregate states. To see this, first consider the simpler case where aggregate productivity
is non-stochastic, so that \( p_t \equiv \bar{p} \) is constant. In this case I seek to show that there exists an equilibrium in which \( \theta \) is constant also. According to (36), this requires that 

\[
x = \frac{\theta r \gamma r}{(1-f(\theta))(1-\eta)}
\]

and 

\[
k^r = \gamma / m(\theta)
\]

are also constant. Specializing (59) to this case and dropping superfluous notation for aggregate productivity shows that

\[
J_j(n) = \max_{n'} \left[ -k^r \max(n' - n, 0) + p \pi_j \bar{y}(n') - n'(1-\eta)(\phi^y + x) + \beta(1-\delta) \sum_{j'=1}^{m_x} \mu(\pi_j, \pi_{j'}) J_{j'}(n') \right].
\] (80)

The right side of (80) is strictly decreasing and continuous in \( \theta \), since both \( x \) and \( k^r \) are continuous and strictly increasing in \( \theta \) while the other terms do not depend on \( \theta \). Moreover, as \( \theta \to 0^+ \), so that \( x, k^r \to 0^+ \), it follows from (??) that \( \sum_{j=1}^{m_x} F^c(\zeta_j) J_j(0) > k^f \). On the other hand, if \( \theta \to +\infty \), then wages become arbitrarily large so that the optimal policy for the firm is to fire its entire workforce. Thus \( J_j(n) \to 0 \) for all \( j \) and all \( n \), so that \( \sum_{j=1}^{m_x} F^c(\zeta_j) J_j(0) \to 0 < k^f \). The intermediate value theorem then implies that there exists a \( \theta \) such that (7) holds, and from the monotonicity in \( \theta \) of the right side of (80) it follows that this solution is unique.

When aggregate productivity is again stochastic, existence of a solution for \( \{\theta_i\}_i \) follows as in the proof of Proposition 5 of Kaas and Kircher (2011), since the transition matrix \( \lambda(p_t, p_{t+1}) \) is strictly diagonally dominant, \( J_{ij}(n) \) is differentiable in \( \theta \), and all elements of the Jacobian \( dJ_{ij}(n)/d\theta_i \) are non-positive. Existence (and uniqueness) of a solution to (60), (61), and (7) then follows from the implicit function theorem in a neighborhood of the non-stochastic solution calculated in the previous paragraph. The only other condition that needs to be checked to ensure that the resulting solution is an equilibrium is that entry should never be negative; however, in the non-stochastic steady state, entry must be constant and strictly positive so as to keep employment constant. Entry after any sequence of realized productivity shocks depends continuously on \( \{z_1, \ldots, z_{m_p}\} \), so it follows by continuity that entry is also always positive provided the aggregate shocks are not too large.

References


