Dynamic Monopsonistic Competition and Labor Market Equilibrium

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Abstract

A tractable variant of the Burdett-Mortensen model of wage setting behavior is formulated and a dynamic market equilibrium solution to the model is defined. In the model, firms cannot commit to wage contracts. Instead, a Markov perfect equilibrium to the wage setting game characterized by Coles (2001) is incorporated into the model. In addition, firm recruiting and hiring decisions, firm entry and exit, and a firm specific productivity shock are included in the specification.

Given that the cost of recruiting workers is proportional to firm employment, we establish the existence of an equilibrium solution to the model that does not require a restriction on the initial distribution of firm sizes. Furthermore, in the special case of homogenous firms, the equilibrium can be described as a solution to two variable system of differential equations. We show that the unique steady state solution is a saddle and that the stable path is the only equilibrium solution.

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1 Introduction

With the publication of a special issue of the *Journal of Labor Economics* on the topic, the concept of *monopsony in the labor market* has achieved a new respectability among labor economists as a formulation of wage setting institutions in labor markets. (See Ashenfelter, Farber, Ransom (2010) for a summary.) In the "new monopsony" literature, the market environment is not the "company town" with a single employer facing a static supply curve as sometimes described in text books. Instead, the source of an employer’s market power is search and matching friction, as in Burdett and Mortensen (1998). In this context, the net flow of workers to a firm is sensitive to its wage relative to that paid by all other firms because price discovery and matching take time. Hence, the market is best described as one of dynamic monopsonistic competition.

Formally, a dynamic monopsonistic equilibrium is a Nash equilibrium solution to a dynamic wage setting game played by many employers in an environment characterized by search friction. In their original formulation of the problem, Burdett and Mortensen (1998) demonstrate that an equilibrium is characterized by wage dispersion defined as different wages paid to identical workers. It is the strategic interaction present in the wage setting environment that contributes to wage dispersion in equilibrium. Understanding this fact is crucial for policy analysis, such as the study of the effects of a minimum wage, as well as for the development of a theory of labor market dynamics.

The model studied in this paper is in the tradition of Diamond (1971), Burdett and Judd (1983), Burdett and Mortensen (1998), Coles (2001) and Moscarini and Postel-Vinay (2010). It differs from these papers by introducing a cost of hiring of a form suggested and estimated by Merz and Yashiv (2007). The purpose of the paper is to introduce a dynamic variant of the Burdett-Mortensen model that can be used for both macro economic and policy applications. Following Coles (2001), the analysis develops the implications of a Markov perfect equilibrium solution to the dynamic wage setting game in which firms cannot precommit to contracts that specify future wages. Interest in this case arises because the Burdett-Mortensen solution is not time consistent.

In the environment studied, there are many workers and firms, indeed a continuum of each. In general, firms have many workers. As is standard in this literature, every agent, worker and firm, is risk neutral and acts to maxi-
mize the expected present value of future income. Workers, employed as well as unemployed, randomly search firms for employment opportunities which arise only sequentially. Each firm chooses a current wage and recruiting effort and hiring strategy and each worker follows a job acceptance and quit strategy that are respectively optimal given rational expectations regarding the future evolution of the market state which characterizes the outcome of their collective behavior. For the specification of the cost of recruiting and hiring considered, one in which the cost is linear in the number of employees, the wage paid and the hiring strategy pursued by any firm are both independent of firm size. This fact provides the simplification needed for tractability without violating the empirical relationships found in micro data.

A equilibrium solution is a market state contingent wage and hiring strategy for each firm and an acceptance and quit strategy for each worker that maximizes each agent’s expected wealth. In the language of Moscarini and Postel-Vinay (2010), a market equilibrium is rank-preserving if the rank of each firm’s lifetime wage offer in the market distribution is its rank in the distribution of productivity at every point in time. In such an equilibrium, turnover is efficient in the sense that every worker moves from a less to a more productive firm whenever an opportunity arise.

Given the restrictions on primitives needed to guarantee the existence of bounded values for all agents in our model, we show that an essentially unique recursive Markov equilibrium exists in the case of equally productive firms. Formally, any equilibrium solution is isomorphic to the unique stable saddle path of a differential equation system that describes the adjustment dynamics of the value of a job-worker match and aggregate employment to their unique steady state values. In the case of general firm heterogeneity with respect to productivity we establish the existence of at least one rank preserving equilibrium.

The paper is most closely related to Moscarini and Postel-Vinay (2010). Although they demonstrate the existence of a recursive rank-preserving equilibrium, restrictions on initial conditions are required in their case. Specifically, because the wage strategy is size dependent in their model, higher paying firms must be larger initially. Unfortunately, this condition is violated in real data because firms die and new firms enter relatively small, independent of productivity. Although this possibility is incorporated in our model, their condition is not needed because a firm’s size does not directly affect the optimal wage choice given the cost of hiring and recruiting function assumed.
Menzio and Shi (2010) develop and study a recursive model of directed search that also allows for search on-the-job. In their paper, they suggest that directed search is a more useful approach for understanding labor market dynamics. They claim that models of random search in the Burdett-Mortensen tradition are intractable because the decision relevant state space is the evolving distribution of wages, which is of infinite dimension. Although the directed search model in arguably simpler in some respects, their principal objection to a random search model is simply not valid in the variant considered in the paper. The shape of the wage offer c.d.f. at any point in time is the equilibrium outcome of the wage and recruiting behavior of firms and the acceptance and quit strategy of worker with a location determined by the distribution of employment over firm types. Hence, at least as an approximation, the decision relevant state variable is a finite vector that characterizes this distribution. Indeed, in the special case of identical firms, the state variable is simply the aggregate level of employment.

2 The Model

2.1 Specification

Time is continuous. The labor market is populated by a unit measure of identical, risk neutral, and immortal workers. Every worker is either unemployed or employed, earns a wage if employed, and the value of home production, \( b \geq 0 \), if not. There is also a measure of risk neutral employing firms. Market output is produced by matched works and firms with a linear technology. New firm enter with frontier technology characterized by labor productivity \( \overline{p} \). Continuing firms are subject to a technology shock process characterized by a given arrival rate \( \gamma \geq 0 \) and a distribution of new values \( \Gamma(p) \) with support \([b, \overline{p}]\) such that \( \lim_{p \uparrow \overline{p}} \Gamma(p) = 1 \). Finally, new firm’s enter at rate \( \mu > 0 \) and continuing firm’s die at frequency \( \delta > 0 \) and the measure of firms is stationary at its steady state value \( \mu/\delta \). As in Klette and Kortum (2005) and Lentz and Mortensen (2008), one can think of the entry flow as firms with new products and the exit flow as firms that are destroyed because their product is no longer in demand.

Let \( p(x) \) represent the productivity of a firm of rank \( x \in [0, \mu/\delta] \) in the distribution of productivity. Given this definition, \( p(x) \) is increasing and its

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\(^1\)These restrictions are made to eliminate special cases of no importance for the analysis.
inverse in the distribution of productivity across firm. As there is persistence in the productivity shock process, a positive measure of firms will all have the highest productivity. Let $\pi$ represent the lowest value of $x$ for which $p(x) = \pi$, the upper support of the distribution of productivity. The value of $\pi$ can is determined by the following argument: Let $F(p)$ represent the faction of firms of productivity $p$ or less. Because new firms enter at the upper support $\pi$ and because a surviving firm experiences a productivity shock, the productivity distribution over firms evolves according to $\dot{F}(p) = \gamma \left( \frac{\gamma}{\delta + \gamma} \right) \Gamma(p) - (\delta + \gamma)F(p)$, for all $p < \pi$. It follows that the steady state fraction of firms with productivity less than $p$ is given by

$$F(p) = \left( \frac{\gamma}{\delta + \gamma} \right) \frac{\mu}{\delta} \Gamma(p)$$

so that $\frac{\delta}{\mu} = \lim_{p \uparrow \pi} F(p) = \frac{\gamma}{\delta + \gamma}$, provided that $\Gamma(p)$ has no mass point at its upper support. As the process converges to the steady state independently of other events in the market, we can simply assume that the steady state represents the distribution of productivity across firms at every point in time. Indeed, formally

$$p(x) = \begin{cases} \Gamma^{-1} \left( \frac{\gamma}{\delta + \gamma} \frac{\delta}{\mu} \right) x & \text{for } x \leq \pi = \left( \frac{\gamma}{\delta + \gamma} \right) \frac{\mu}{\gamma} \\ \pi & \text{for } \pi < x < \frac{\mu}{\delta}. \end{cases}$$

At any point in time, an operating firm is described by its productivity rank $x$, the size of its labor force $n$, and market state variable $s$. We assume that the market state process $\{s_t\}$ is deterministic and is know to all agents and exogenous to the behavior of any one of them. Subsequently, we will derive the information relevant state from the structure of the model.

A firm offers the wage $w(x, n, s)$ to all applicants at any point in time. Let $W(x, n, s)$ denote a worker’s associated expected lifetime wage when employed by such a firm and let $U(s)$ denote the value of being unemployed in market state $s$. Define $G(W, s)$ as the proportion of workers who enjoy lifetime value at least as great as $W$ in aggregate state $s$. As no employee finds $W < U(s)$ acceptable, $G(U, s)$ describes the unemployment rate in state $s$. For $W \geq U(s)$, $G(W, s) - G(U(s), s)$ is the proportion of workers who are employed a lifetime wage $W$ or less.

New firms enters with a single worker, the innovator. Once a new firm enters, its creator sells the firm to risk neutral investors for its value and
reverts to his or her role as a worker. However, each continuing firm faces costs of expanding its labor force. It does so by posting a number of vacancies, denoted as $Y$. As workers only quit to either create a firm or earn a higher lifetime wage, a continuing firm paying $W$ that posts $Y$ vacancies hires at expected rate $H = \eta(s)G(W, s)Y$ where $\eta(s)$, the frequency with which a vacancy contacts one worker and $G(W, s)$ is the fraction of all the workers that are earning less than $W$. In other words, $\eta(s)Ydt$ describes the probability that a potential hire is contacted over the next instant $dt$ and $G(W, s)$ is the probability that any randomly contacted worker is willing to accept the job. Of course, the firm’s employees quit at rate $q(W, s) = \mu + \lambda(s)[1 - F(W, s)]$ where $\mu$ is the frequency with which a worker creates a new firm, $\lambda(s)$ is the offer arrival rate, and $F(W, s)$ is the probability that an offer is less than or equal to $W$. Of course, this specification implies that workers always prefer to quit to create a new firm relative to the option of forgoing the opportunity and remaining employed with the old one, a condition which holds under the assumption that new firms are the most productive.

There are two costs of acquiring additional employees. Let $k$ represent the per unit cost of posting a vacancy. Perhaps more importantly, vetting job applicants and training new hires is also costly in the real world. The total flow recruitment and hiring cost take the form $(ky + c(h))n$ where $y = Y/n$ is the number of vacancies posted per employee and $h = H/n$ represents the expected hire rate per employee and $c(h)n$ denotes hiring costs. Note that by scaling cost by employment, each employee is implicitly involved in the training process which is otherwise subject to increasing costs at the margin.\(^2\) Because hiring requires recruiting effort, the expected hire rate per employee is $h(W, s) = H/n = \eta(s)G(W, s)y$. 

### 2.2 The Wage Setting Game

A central feature of the analysis in this paper is a firm’s inability to commit to future wages. Why does a firm pay any more than the worker’s reservation wage? Coles (2001) shows how to construct a subgame perfect (Markov) equilibrium to the original Burdett and Mortensen (1998) framework in which firms do pay more. As the formal arguments are detailed in Coles (2001), here we skip some of the more tedious technical details.

\(^2\)This formulation of costs of adjustment is standard in the literature. As a case in point, see Merz and Yashiv (2007).
Of course, a specification of a subgame perfect equilibrium requires a
description of behavior off the equilibrium path. The issue is particularly
interesting in the Burdett-Mortensen framework. Specifically, suppose the
firm \((x, n)\) deviates from the equilibrium wage \(w(x, n, s)\) by announcing wage
\(w = 0\). How should employees respond? If employees anticipate the firm will,
after a small delay of length \(dt\), return to the equilibrium wage strategy, then
the wage deviation will have (almost) no effect on the expected present value
of the worker’s future earnings and thus there will be no turnover response.
But if turnover does not respond to the wage deviation, then announcing
\(w = 0\) is a profitable deviation.

Hence, if \(w(x, n, s)\) describes an optimal equilibrium wage strategy, then
subgame perfection requires a credible “punishment strategy” by workers to
any deviation. But with a Markov equilibrium, the Gordian knot is that
it is always an optimal strategy for the firm to return to the equilibrium
path. If the firm does so, it is not then individually rational for workers to
change their turnover choices. Coles (2001) resolves this puzzle by identify-
ing a Markov equilibrium to the wage choice subgame in which the firm is
everywhere indifferent between two pricing strategies: Sticking to the high
wage path (always announce \(w = w(x, n, s)\)) and forever paying the workers’
common reservation wage, denoted as \(R\). The following actions (and off the
equilibrium path beliefs) support the higher wage along an equilibrium path:

(a) Should the firm post the deviating wage \(w^d < R\) then, in the future,
the firm posts wage \(w_0 = R\). As all employees prefer to quit and no one
accepts employment in the future, announcing \(w^d < R\) yields zero profit.

(b) Should the firm post \(w^d \in (R; w(x, n, s))\), then in the future the firm
posts wage \(w_0 = R\). Employees do not quit into unemployment but each will
quit to the first outside offer (as they anticipate \(w_0 = R\) in the entire future).
Given this turnover response, announcing \(w^d = R\) dominates posting \(w^d\)
as the higher deviating wage retains no additional employees and is costly.

(c) Should the firm post \(w^d > w(x, n, s)\), then in the future the firm
always posts wage \(w_0 = w(x, n, s)\). As this wage deviation does not change
expected future wages, worker turnover is unchanged. Announcing \(w(x, n, s)\)
dominates posting \(w^d\) because again a higher deviating wage retains no extra
employees and is costly.
2.3 The Reservation Wage

Because individual workers are hired and quit sequentially, the number of employees in a continuing firm is a stochastic process. Indeed, $n$ is a birth-death process with an absorbing state that occurs when the firm dies. That is, the firm’s labor force size is an integer that can only transit from the value $n$ to $n+1$ if a worker is hired, from $n$ to $n-1$ if a worker quits, or to zero if the firm loses its market in any sufficiently short time period of length $dt > 0$.

The transition rates for these three events are respectively the hire frequency, the quit frequency, and the destruction frequency, $h(W,s)n$, $q(W,s)n$, and $\delta$.

Consider a worker in a firm characterized by $(x,n,s)$ that pursues the high wage strategy $w(x,n,s)$. The value of employment or lifetime wage $W(x,n,s)$ solves the first order differential equation

$$rW(x,n,s) = w(x,n,s) + \delta[U(s) - W(x,n,s)]$$

$$+ \frac{\gamma}{\bar{x}} \int_{0}^{\bar{x}} [\max \langle W(z,n,s), U(s) \rangle - W(x,n,s)] \, dz$$

$$+ \lambda(s) \int [\max \langle X, W(x,n,s) \rangle - W(x,n,s)] \, dF(X,s)$$

$$+ \frac{\mu}{\mu/\delta - \bar{x}} \int_{\bar{x}}^{\mu/\delta} ((\Pi(z,1,s) + W(z,1,s)dx - W(x,n,s)) \, dx$$

$$+ q(W(x,n-1,s),s)(n-1)[W(x,n-1,s) - W(x,n,s)]$$

$$+ h(W(x,n-1,s),s)n[W(x,n+1,s) - W(x,n,s)] + \frac{\partial W}{\partial t}.$$ 

where $\Pi(x,n,s)$ is the value of a firm of rank $x$ with $n$ employees in state $s$. In other words, the flow value of employment is equal to the wage income plus the expected capital gain or loss associated with the possibility of firm destruction, a productivity shock, creating a firm, finding a better job, a quit by some other employee, a new hire, and the passage of time all conditional on the market state $s$. The fact that productivity rank is reassigned uniformly over the interval $[0,\bar{x}]$ in response to a productivity shock is implied by the fact that $d\Gamma(p(x)) = \Gamma'(p(x))p'(x)dx = dx/\bar{x}$ from (1). Finally, the productivity rank of a new firm, $x \in (\bar{x}, \mu/\delta]$, is also assigned randomly to distinguish among the firms that are equally productive.

Should the employer deviate by paying the worker’s reservation wage, the worker expects that that same wage will be paid in the future even if the
firm productivity experiences productivity shock. Consequently,

\[ rW^d(x, n, s) = R(x, n, s) + \delta[U(s) - W^d(x, n, s)] \]

\[ + \lambda(s) \int \left[ \max \langle X, W^d(x, n, s) \rangle - W^d(x, n, s) \right] dF(X, s) \]

\[ + \frac{\mu}{\mu/\delta - \overline{x}} \int_{\overline{x}}^{\mu/\delta} (\Pi(z, 1, s) + W(z, 1, s)dx - W(x, n, s)) dx \]

\[ + q(W^d(x, n - 1, s), s)(n - 1)[W^d(x, n - 1, s) - W^d(x, n, s)] \]

\[ + h(W, s)n[W^d(x, n + 1, s) - W^d(x, n, s)] + \frac{\partial W}{\partial t}. \]

For an unemployed worker, the value of unemployment \( U(s) \) solves the analogous equation

\[ rU(s) = b + \lambda(s) \int \left[ \max \langle X, U(s) \rangle - U(s) \right] dF(X, s) \]

\[ + \frac{\mu}{\mu/\delta - \overline{x}} \int_{\overline{x}}^{\mu/\delta} (\Pi(z, 1, s) + W(z, 1, s)dx - W(x, n, s)) dx \]

\[ + \frac{\partial U}{\partial t}. \]

By definition, the reservation wage for any firm is the value of the wage that equates the value of employment in the firm to the value of unemployment. This equality, \( W^d(x, n, s) = U(s) \) for all \((x, n, s)\) implies that the reservation wage is the solution to the value of home production

\[ R(x, n, s) = b \]

in all circumstances.

### 2.4 The Size Independent Wage and Recruiting Policy

In this section, we verify that a size independent equilibrium optimal wage and recruiting policy exist. We do so by showing that a size independent policy is optimal for a firm if all others pursue such a policy. In the process, we also characterize the unique size independent wage and recruiting policy.

Let \( W(x, s) \) represent the value of employment to a worker given that all firms pursue size independent wage and recruiting policies in the future. In this case, the reservation wage, \( R(s) \) is independent of both firm size and
productivity from (4). Consider a potential firm that pays wage \(w(x, s)\) along the equilibrium path. In the case of an operating firm with \(n \geq 1\) employees the value of this firm, denoted \(\Pi(x, n, s)\), is given by the Bellman equation

\[
 r\Pi(x, n, s) = \max_{y \geq 0} \left\{ np(x) - w(x, s) \right\}
\]

\[
 -\delta \Pi(x, n, s) + \frac{\alpha}{\pi} \int_0^\pi (\Pi(z, n, s) - \Pi(x, n, s)) \, dz
\]

\[
 + \eta(s)G(W(x, s), s) \, y \, n \, [\Pi(x, n + 1, s) - \Pi(x, n, s)]
\]

\[
 + q(W, s) \, n \, [\Pi(x, n - 1, s) - \Pi(x, n, s)]
\]

\[
 + \mathbb{E} \left[ \frac{\partial \Pi(x, n, s)}{\partial t} \right] |s_t = s]
\]

where the right hand side is largest value of the current cash flow plus the capital gain or loss associated with the destruction of the firm, a productivity shock, adding a worker, losing a worker, and the passage of time respectively. As noted above, an equilibrium with no wage commitment requires that the firm is indifferent between continuing to announce the equilibrium wage policy and announcing the reservation wage \(R = b\). So, the value of the deviating strategy is defined analogously by

\[
 r\Pi^d(x, n, s) = \max_{y^d \geq 0} \left\{ np(x) - b - y^dnk - nc(\eta(s)G(U(s), s)y^d)
\right\}
\]

\[
 -\delta \Pi^d(x, n, s) + \frac{\alpha}{\pi} \int_0^\pi (\Pi^d(z, n, s) - \Pi^d(x, n, s)) \, dz
\]

\[
 + \eta(s)G(U(s), s)\gamma \, n \, [\Pi^d(x, n + 1, s) - \Pi^d(x, n, s)]
\]

\[
 + q(U(s), s) \, n \, [\Pi^d(x, n - 1, s) - \Pi^d(x, n, s)]
\]

\[
 + \mathbb{E} \left[ \frac{\partial \Pi^d(x, n, s)}{\partial t} \right] |s_t = s]
\]

Note that the solution to the second equation is of the proportional form \(\Pi^d(x, n, s) = \nu^d(x, s)n\) where \(\nu^d(x, s)\) solves

\[
 (r + \delta + \gamma)\nu^d(x, s) = \max_{y^d \geq 0} \left\{ \frac{p(x) - b - y^dk - c(\eta(s)G(U(s), s)y^d)}{\nu^d(x, s)} + \mathbb{E} \left[ \frac{\partial \nu^d(x, s)}{\partial t} \right] |s_t = s] \right\}.
\]

Furthermore, the fact that the high wage policy is such that \(\Pi^d(x, n, s) = \Pi(x, n, s)\) for all \((x, n, s)\) implies that the wage is independent of size. Specifically, \(\Pi(x, n, s) = v(x, s)n\) where

\[
 (r + \delta + \gamma)v(x, s) = \max_{y^d \geq 0} \left\{ \frac{p(x) - w(x, s) - yk - c(\eta(s)G(W(x, s), s)y)}{v(x, s)} + \mathbb{E} \left[ \frac{\partial v(x, s)}{\partial t} \right] |s_t = s] \right\}.
\]
Finally, because \( v(x, s) = v^d(x, s) \),

\[
 w(x, s) = b + [q(U(s), s) - q(W(x, s), s)]v(x, s) 
\]

(7)

\[
 \max_{y \geq 0} \left[ -yk - c(\eta(s)G(W(x, s), s)y) + \eta(s)G(W(x, s), s)yu(x, s) \right] 
\]

\[
 - \max_{y^d \geq 0} \left[ -y^dk - c(\eta(s)G(U, s)y^d) + \eta(s)G(U, s)y^dv(x, s) \right]. 
\]

In other words, the wage offered is equal to the reservation wage plus the saving in the flow cost of turnover per worker attributable to paying the higher wage, all of which are independent of the firm’s size.

### 2.5 A Baseline Case

The model simplifies considerably if the cost of posting a vacancy, \( k \), is trivial. The cost of adjustment function estimated by Merz and Yashiv (2007) provides empirical support for the relevance of this case. In the notation of this model, they estimate a cost of recruiting and hiring function of the form \((ah + c_0h^c_1)n\), which is a special case of that specified above since the hire rate is linear in the number of vacancies posted. Although the estimates of the non-linear component play a large role in explaining the dynamics of factor adjustment as well as the ability of their model to explain firm market values, their point estimate of the linear component is slightly negative and statistically insignificant. Hence, the restriction \( k = 0 \) is consistent with their findings. Furthermore, this evidence that labor adjustment costs are primarily processing and training costs rather than those associated with advertising and recruiting seems to be consistent with everyday experience.

When \( k = 0 \), the optimal equilibrium and deviating recruiting policies are identical, i.e., \( y = y^d \). Consequently, the high wage policy in this case reduces to

\[
 w(x, s) = b + [q(U(s), s) - q(W(x, s), s)]v(s, s) 
\]

(8)

by equation (4), \( F(U(s)) = 0 \), and \( q(W, s) = \mu + \lambda(s)[1 - F(W, s)] \). In other words, the wage premium is equal to the saving in the flow cost of quits attributable to paying the higher wage. Finally, because all equilibrium offers are acceptable, the hire frequency can be regarded as the choice variable when
\( k = 0 \). Hence, the value of a worker to a firm of rank \( x \) is the solution to the first order differential equation

\[
(r + \delta + \gamma)v(x, s) - E\frac{\partial v}{\partial t} = s
\]

\( = \max_{h \geq 0} \left\{ p(x) - b + \frac{\gamma}{\pi} \int_0^\pi v(z, s) dz - c(h) \right\}. \]

The analysis presented in the rest of the paper is restricted to this case.

3 Rank Preserving Equilibria

3.1 Comparative Dynamics

A wage policy function \( w(x, s) \) that is strictly increasing in \( x \) for every state of the market \( s \), is said to be rank-preserving in the sense that the rank of a firm’s wage in the wage offer distribution is the same as its rank in the productivity distribution in every market state. Because workers voluntarily move only from less to more productive firms in this case, a market equilibrium is rank-preserving in the sense of Moscarini and Postel-Vinay if the equilibrium wage policy function is rank preserving.

Equation (8) implies that the wage policy function is rank preserving in productivity rank if the value of employment and the value of a match are increasing in productivity rank. The results of this section establish the converse.

We need to establish that the life time wage has the property that more productive firms offer a higher life time wage in any equilibrium if more productive firms pursue a more generous wage policy and that the value of a worker is higher in such a firm. From equations (2) and (3),

\[
rW(x, s) = w(x, s) + \delta[U(s) - W(x, s)] + \frac{\gamma}{\pi} \int_0^\pi \left[ \max \langle W(z, s), U(s) \rangle - W(x, s) \right] dz
\]

\[
+ \lambda(s) \int_{W(x, s)}^W \left[ \max \langle X, W(x, s) \rangle - W(x, s) \right] dF(X, s)
\]

\[
+ \frac{\mu}{\mu/\delta - \pi} \int_\pi^{\mu/\delta} [v(z, s) + W(z, s) - W(x, s)] dz + \frac{\partial W}{\partial t}
\]
and

\[ rU(s) = b + \lambda(s) \int_U^W [\max \langle X, U(s) \rangle - U(s)] dF(X, s) + \frac{\mu}{\mu/\delta - x} \int_\tau^\infty [v(z, s) + W(z, s) - U(x, s)] dz + \frac{\partial U}{\partial t}. \]

Hence, equation (8) implies

\[ (r + \delta + \mu + \gamma + \lambda(s)[1 - \hat{F}(x, s)])\sigma(x, s) = W(x, s) - U(s) \text{ is unique, positive, continuous and increasing in } x. \]

**Proposition 1** If \( w(x, s) \) is continuous and increasing \( x \) for every \( s \), then the surplus value of employment, \( W(x, s) - U(s) \) is unique, positive, continuous and increasing in \( x \).

**Proof.** Any forward solution to equation (10) is a fixed point of the continuous and increasing contraction \( T \) defined by

\[ (T\sigma)(x, s) = E_t \left\{ \int_t^\infty \left( w(x, s_\tau) - b + \gamma \int_0^x \sigma(z, s_\tau) dz - \lambda(s) \int_0^x \sigma(z, s_\tau) \sigma'(z, s_\tau) dz + \frac{\partial \sigma}{\partial t} \right) e^{-\int_t^\tau (r + \delta + \mu + \gamma + \lambda(s_\tau)) d\tau} d\tau \right\} \]

where \( E_t \) denotes the expectation with respect to information available at time \( t \). Obviously, \( T \) maps the set of bounded, increasing and continuous functions of \( x \) into itself under the hypothesis. Hence, any fixed point of the map is unique has the property.

**Proposition 2** The value of a match to a firm \( v(x, s) \) is unique, continuous and increasing in \( x \) for every \( s \).

**Proof.** Any solution \( v(x, s) \) to (9) is a fixed point of the contraction

\[ (Tv)(x, s) = E_t \left\{ \int_t^\infty \left( p(x) - b + \gamma \int_0^x v(z, s_\tau) dz + \max_{h \geq 0} \{ hv(x, s_\tau) - c(h) \} \right) e^{-\int_t^\tau (r + \delta + \mu + \gamma + \lambda(s_\tau)) d\tau} d\tau \right\} \]

Given that \( (Tv)(x, s) \) is continuous and increasing in \( x \) if \( v(x, s) \) has these properties, \( T \) maps the space of bounded, continuous and increasing functions in \( x \).
3.2 Aggregation

A market equilibrium requires that the offer distribution $F(X, s)$ and the offer arrival rate $\lambda(s)$ are consistent with the wage and recruiting strategies of all the firms. We restrict attention to the case of strict heterogeneity for the moment, i.e., $p(x)$ strictly increasing. The homogeneous firm case or more generally one in which the productivity distribution contains mass points considered subsequently can then be regarded as the limiting case of $p(x) \to p$ for all $x$ on some interval. In the analysis that follows, we suppose that the optimal wage policy function is rank-preserving and confirm the existence and uniqueness of such a policy as part of the characterization of the equilibrium.

To illustrate the endogeneity of the offer distribution and arrival rate, one needs to aggregate across the firms. At a point in time, the firm of rank $x$ has $n(x)$ employees defined on the integers. Hence, $N(x) = \int_0^x n(x) dx$ in the limiting case of a continuum of firms by the law of large numbers.

The fact that every hire is an acceptable match is expressed by the following equality:

$$H(x) = \eta(s) G(W(x, s)) Y(x, s) = h(v(x, s)) n(x)$$

Recall that $Y(p(x), s)$ is the number of vacancies posted by the firm of rank $x$ and

$$G(W(x, s)) = 1 - N + N(x)$$

is the expected fraction of applicants that accept offers by firms of rank $x$ or less where $N = N(1)$ represents aggregate employment. Therefore, in the limiting case of a continuum of firms, the expression

$$\int_0^x \eta(s) G(W(x, s)) Y(p(x), s) dz = \int_0^x h(v(z, s)) dN(z) \forall x \in (0, 1)$$

holds.

Because every meeting involves one worker and one vacancy, the two meeting rates are related by the matching identity

$$\lambda(s) \equiv \eta(s) \int_0^1 Y(p(x), s) dx.$$ 

As the offer distribution is the distribution of vacancies over firms by definition,

$$\hat{F}(x, s) \equiv F(W(p(x), s), s) = \frac{\int_0^x Y(p(z), s) dz}{\int_0^1 Y(p(z), s) dz}.$$
these equations imply
\[ \int_0^x \lambda(s) (1 - N + N(z)) d\hat{F}(z, s) = \int_0^x h(v(x, s)) dN(z), x \in [0, 1]. \]

As a consequence, the offer distribution solves
\[ \lambda(s)\hat{F}(x, s) = \int_0^x \frac{h(v(z, s))dN(z)}{1 - N + N(z)} \] (11)

where the offer arrival rate is
\[ \lambda(s) = \int_0^1 \frac{h(v(z, s))dN(z)}{1 - N + N(z)}. \] (12)

### 3.3 Definition

Because all new firms are of the highest productivity, the change in employment of firms of rank \( x \) or less is the flow into the set from unemployment plus those employed in firms that receive an appropriate productivity shock less the sum of the flows associated with firm destruction, firm creation, productivity shocks, and quits to more productive firms. Formally, for low values of \( x \)
\[ \hat{N}(x) = \lambda(s)\hat{F}(x, s) (1 - N(1)) \]
\[ - \left( \delta + \mu + \gamma \left[ 1 - \frac{x}{\bar{x}} \right] + \lambda(s) [1 - \hat{F}(x, s)] \right) N(x), \] (13)

\( x \in [0, \bar{x}) \), where \( \bar{x} \) is the constant specified in (1). As new firms enter with a single worker and the rank index is uniformly distributed over the interval \([\bar{x}, \mu/\delta]\),
\[ \hat{N}(x) = \frac{(x - \bar{x})\mu}{\mu/\delta - \bar{x}} + \lambda(s)\hat{F}(x, s) (1 - N) \]
\[ - \left( \delta + \mu + \gamma + \lambda(s) [1 - \hat{F}(x, s)] \right) N(x), \] (14)

\( x \in [\bar{x}, \mu/\delta] \) where the first term is the flow of new entrants and \( N \equiv N(\mu/\delta) \) is aggregate employment.

Equations (11) and (12) imply that the distribution of employment over firm rank is the information relevant sufficient statistic for the market state; i.e.
\[ s \in \{ N : [0, \mu/\delta] \to [0, 1] | N(x) \text{ is increasing and continuous in } x \}. \] (15)
In other words, the market state \( \{ s_t \} \) is a degenerate process characterized by the system of first order differential equations defined in (13) and (14), its laws of motion.

**Definition** A stationary recursive *rank preserving monopsonistic equilibrium* is a wage policy function \( w(x; N) \) that is increasing in \( x \), hire rate policy \( h(v(x, N)) \), a reservation wage \( R(N) \), and a value of a worker for each firm \( v(x, N) \) that satisfy

\[
w(x, N) = R(N) + \lambda(N) \hat{F}(x, N)v(x, N), x \in [0, \mu/\delta], \tag{16}
\]

\[
h(v(x, N)) = \max_{h \geq 0} \{ hv(x, N) - c(h) \} \tag{17}
\]

\[
R(N) = b \tag{18}
\]

such that \( v_t(x) = v(x, N_t) \) and \( N_t(x) \) for all \( x \) is a solution to the system of ordinary differential equations composed of equations (13) and (14) together with

\[
(r + \delta + \mu + \gamma + \lambda)v(x) - \dot{v}(x) = p(x) - b + \frac{\gamma}{\delta} \int_0^x (v(z) + \sigma(z)) \, dz + \max_{h \geq 0} \{ hv(x) - c(h) \} \tag{19}
\]

where

\[
\lambda \hat{F}(x) = \int_0^x \frac{h(v(z)) \, dN(z)}{1 - N + N(z)}. \tag{20}
\]

Finally, an equilibrium is a particular solution to the system consistent with the initial distribution of employment \( N_0 : [0, \mu/\delta] \to [0, 1] \) and the transversality conditions

\[
\lim_{t \to \infty} v(x) e^{-rt} \, dt = 0. \tag{21}
\]

**Assumption 1:** The hire rate function \( h(v) \) defined by equation (17) exists and is Lipschitz continuous in \( v \).

**Assumption 2:** A positive solution for \( \bar{v} \) exists to

\[
\bar{v} = \max_{h \geq 0} \left\{ \bar{p} - b + \max_{h \geq 0} \left\{ h\bar{v} - c(h) \right\} \right\}, \tag{22}
\]

16
Assumption 1 is required to guarantee that the differential equations that characterize an equilibrium have a unique solution for any specification of initial conditions. In the proofs that follow, Assumption 2 is used to show that a stationary solution exists which also satisfies the transversality condition for the following reason:

**Proposition 3** In any recursive rank-preserving equilibrium, the hire rate is bounded above by the effective rate of time discount, i.e., \( h(v) < r + \delta + \mu + \gamma \) for all \( v \leq \bar{v} \).

**Proof.** As \( \bar{v} = \max_{h \geq 0} \left( \frac{\bar{p} - b - c(h)}{r + \delta + \mu + \gamma - h} \right) \) the assertion follows as a corollary of Assumption 2. □

### 3.4 Homogenous Firms

Although it is true that the market state space is of infinite dimension in the general case, it need not be so in practice. In this section, a unique recursive rank-preserving equilibrium solution to the model is shown to exist for the case of homogenous firms. Furthermore, the market state is aggregate employment, a scalar.

In the homogeneous case, \( \gamma = 0 \) which implies that \( p(x) = \bar{p} \) for all \( x \), \( \bar{p} = 0 \) from equation (1), and \( R(s) = b \) from equation (18). Note that

\[
\lambda(s) = h(v(s)) \int_{0}^{1} \frac{dN(z)}{1 - N(1) + N(z)} = h(v(s)) \ln \left( \frac{1}{1 - N} \right)
\]

from (12) where \( v(s) = v(x, s) \) is the common value of a worker in all the firms. Hence, the state is simply aggregate employment which evolves according to

\[
\dot{N} = \mu + h(v) \ln \left( \frac{1}{1 - N} \right) (1 - N) - (\delta + \mu) N
\]

(24)

by (14) where the common value of a match is a solution to

\[
\dot{v} = \left( r + \delta + \mu + h(v) \ln \left( \frac{1}{1 - N} \right) \right) v - \left( \bar{p} - b + \max_{h \geq 0} \{ hv - c(h) \} \right).
\]

(25)
As the solution of interest, $v(N)$, is the stationary equilibrium value of a job-worker match when aggregate employment is $N$, it solves the differential equation

$$\frac{dv}{dN} = \frac{\left( r + \delta + \mu + h(v) \ln \left( \frac{1}{1-N} \right) \right) v - (\overline{p} - b + \max_{h \geq 0} \{hv - c(h)\})}{\mu + h(v) \ln \left( \frac{1}{1-N} \right) (1 - N) - (\delta + \mu) N}.$$ 

It is well known that a unique continuous solution exists to this equation for all $N \in [0, 1]$ if and only if the ODE system composed of (24) and (25) has a unique steady state solution and the steady state is a saddle point. Indeed, the branch of the saddle path that converges to the steady state for every initial value of aggregate employment is the equilibrium value of a match function. Below we prove that these necessary and sufficient conditions holds.

Any steady state solution to the differential equation system is the $(N, v)$ pair implicitly defined by the system of equations

$$(\delta + \mu) \left( N - \frac{\mu}{\delta + \mu} \right) = h(v) \ln \left( \frac{1}{1-N} \right) (1 - N)$$

(26)

$$\left( r + \delta + \mu + h(v) \ln \left( \frac{1}{1-N} \right) \right) v = \overline{p} - b + \max_{h \geq 0} \{hv - c(h)\}$$

(27)

There exists a single solution pair $(v, N)$ to these equations.

The LHS of (26) is zero at $N = \frac{\mu}{\delta + \mu} < 1$ and increases at the constant rate $\delta + \mu$ while the RHS is zero at $N = 0$, and is increasing, strictly concave, and tends to 0 as $N$ approaches 1. Hence, a unique positive value of $N$ strictly less than 1 exists for every positive value of $v$. Furthermore, the solution for $N$ increases with $v$ given that $h(v)$ is increasing. As the LHS of (27) is positive and strictly increasing in $N$ and the RHS is independent of $N$ and positive given $\overline{p} > b$, at most one value of $N$ exists that satisfies the equation for every positive finite value of $v$. As the RHS is positive when $(v, N) > 0$ and is increasing in $v$ at a rate less than $r + \delta + \mu$ by the envelope theorem and the fact that $\overline{h}(N) < r + \delta + \mu$, the implicit relationship between $v$ and $N$ defined by equation (27) is decreasing. Hence, a unique positive state solution for the pair $(v, N)$ exists and $N < 1$.

The dynamics implied by the ODE system composed of (24) and (25) are illustrated by its phase diagram portrayed in Figure 1. As the RHS of (25) is increasing in $v$, $\dot{v} > (\dot{v})0$ above (below) the $\dot{v} = 0$ curve as indicated by
the directional arrows on the solution trajectories in Figure 1. Because the RHS of (24) is increasing in $v$, $\dot{N} > (\leq) 0$ to the above (below) the $\dot{N} = 0$ curve. Hence, the intersection of the two singular curves is a saddle point that attracts a unique converging saddle path from any initial value of $N$. Finally, because the growth rate in $v$ on the unstable path above the steady state exceeds the rate of interest, the stable path represents the only Markov equilibrium.

**Proposition 4** A unique stationary recursive monopsonistic rank-preserving equilibrium exists in the limiting case of equally productive firms.

The phase diagram and the equilibrium adjustment path have a natural interpretation as a dynamic supply and demand model for match formation. Of course, the match value $v$ is the relevant price. A higher value of a match induces recruiting effort which yields greater utilization of the available labor force by reducing unemployment duration, a relationship represented by the upward sloping $\dot{N} = 0$ singular (supply) curve. However, the (demand) price $v$ falls at the margin with $N$ along the $\dot{v} = 0$ curve because life time wages
increase with aggregate employment. Along the optimal adjustment path, the price adjusts in response to "excess demand" to bring the supply and demand prices into balance.

Note that any common and unanticipated positive shock to the productivity of a match \( p \) shifts the dynamic "demand" curve to the right in Figure 1 but has no direct effect on "supply." The result is an increase in the steady state values of both employment and the surplus value of a match as in the canonical search and matching model. Hence, the equilibrium value of \( v \) jumps up initially and then adjusts slowly downward along the path converging to the new steady state value.

### 3.5 Heterogeneous Firms

As demonstrated above, the structure of the model is quite simple in the limiting case of equally productive firms. The wage distribution is determined as a function of the only decision relevant state variable, the level of aggregate employment. The purpose of this section is to show that this structure generalizes in the case of heterogeneous firms when there are a finite number of firm types.

Let \( p_i \) represent the productivity of all firms of type type \( i = 1, \ldots, I \). Therefore, \( p(x) = p_i \) for all \( x \in (x_{i-1}, x_i] \) where the \( (x_{i-1}, x_i] \) represents the set of firms of type \( i \) where \( x_0 = 0 \). As the value of a worker is the same for all firms of the same type, \( v_i = v(x_i) \) and \( x \in (x_{i-1}, x_i] \), \( i = 1, 2, \ldots I \), denotes the common value of a worker to a firm and surplus value of employment in firm of rank \( k \). Given this notation, the state is a vector \( \mathbf{N} = (N_1, N_2, \ldots, N_I) \) where \( N_i = N(x_i) \) and represents employment of firms of type \( i \) or less. In this case, equation (20) implies

\[
\lambda_i(\mathbf{v}, \mathbf{N}) = \lambda \hat{F}(x_i) = \sum_{j=1}^{i} h(v_j) \int_{x_{j-1}}^{x_j} \frac{dN(z)}{1 - N + N(z)}
\]

where \( N = N_I \) is measure of aggregate employment and vectors \( \mathbf{v} = (v_1, \ldots, v_I) \) and \( \mathbf{N} = (N_1, \ldots, N_I) \) represent the values of the workers across types and the employment distribution over types respectively.
Obviously, equations (19) and (28) can be written as

\[ \dot{v}_i = (r + \delta + \mu + \gamma + \lambda(v, N))v_i \]

\[ - \left( p_i - b + \frac{\gamma}{I-1} \sum_{j=1}^{I-1} (v_j + \sigma_j) + \max_{h \geq 0} \langle hv_i - c(h) \rangle \right) \]

The dynamics of the distribution of employment over types are determined by the following differential equations:

\[ \dot{N}_i = \lambda_i(v, N) (1 - N) - \left( \delta + \mu + \gamma \left[ 1 - \frac{x_i+1}{\bar{x}} \right] \right. \]

\[ + \left[ \lambda(v, N) - \lambda_i(v, N) \right] \right) N_i, \quad (30) \]

\[ i = 1, ..., I - 1 \] and \( x_I = \bar{x} \). Finally, the law of motion for aggregate employment, \( \dot{N} = N_t \), is

\[ \dot{N} = \mu + \lambda(v, N) (1 - N) - (\delta + \mu + \gamma) N \]

Assumption 1 and equation (28) imply that the functions \( \lambda_i(v, N), i = 1, ..., I \), are all Lipschitz continuous in \( v \). As \( \ln(x) \) is differentiable and has bounded derivatives on \( x > 0 \), the right hand sides of the system of ODE composed of equations (29), (30), and (31) are Lipschitz continuous for all \( N \) that satisfy

\[ N < 1. \]

because \( N_i \leq N \) for all \( i \). Hence, the system of ODE has a unique solution for every set of boundary conditions.

As a stationary equilibrium, we seek a particular solution to the system that can be represented as a function \( v(N) \) that map the set of employment distributions into the feasible set of value vectors given that map is used to solve for the evolution of the distribution of employment. An equilibrium, then, is a fixed point of the transformation \( M \) defined by the forward solutions to (29), expressed here as

\[ (Mv)_i(N_0) = \int_0^\infty \left( p_i - b + \frac{\gamma}{I-1} \sum_{i=1}^{I-1} (v_i(N_t) + \sigma_i(N_t)) \right) \]

\[ \times e^{-\int_0^t (r+\delta+\mu+\gamma+\lambda(v(N_t),N_t))dt} \]

where \( N_t \) is the backward solution to the laws of motion for the employment distribution, equations (30) and (31), conditional on the given initial
distribution of employment over firm types. That is

\[(MN)_t (N_0) = N_0 e^{-\int_0^t \alpha_i(N_\tau) d\tau} + \int_0^t \lambda_i(v(N_\tau), N_\tau) (1 - N_\tau) e^{-\int_0^\tau \alpha_i(N_z) dz} d\tau,\]

\[\alpha_i(N) = \left( \delta + \mu + \gamma \left[ 1 - \frac{\tau_i}{T} \right] + [\lambda(v(N), N) - \lambda_i(v(N), N)] \right), i = 1, ..., I - 1,\]

and

\[(MN)_t (N_0) = N_0 e^{-(\delta + \mu + \gamma)t} + \int_0^t \lambda(v(N_\tau), N_\tau) (1 - N_\tau) e^{-(\delta + \mu + \gamma)(t - \tau)} d\tau.\]

As \(M\) maps the set of positive and Lipschitz continuous vector functions into itself, it has a fixed point that is bounded in the limit as \(t \to \infty\) by Schauder’s theorem if it also maps a subset of functions that are bounded in the supnorm into itself. Furthermore, any fixed point is an recursive stationary equilibrium because it satisfies the transversality condition.

**Proposition 5** In the case of any finite number of firm types distinguished by productivity, a stationary recursive monopsonistic rank-preserving equilibrium exists if \(N_0 < 1\).

**Proof.** To complete the proof, we need only show that \(M\) maps a subset of functions bounded in supnorm into itself. Suppose that \(v_i(N_i) \leq \overline{v}\) and \(N_i \leq \overline{N}\) for all \(i\) where \(\overline{v} < \infty\) and \(\overline{N} < 1\) are positive and finite constants to be determined. Let \(\|f\| = \sup_{f \in F} |f|\) represent the supnorm on the set of functions Lipschitz continuous function \(F\). As \(\lambda_i(v_i, N_i) \leq \lambda(v_i, N_i)\) for all \(i\), and \(0 \leq MN_i \leq MN_i\) if \(0 \leq N_i \leq N_i\), it is sufficient to show that \(MN_i \leq \overline{N} < 1\) and \(Mv_i \leq \overline{v}\).
If \( v_i(N) \leq \bar{v} \) for all \( i \), then

\[
\| (MN)_t (N_0) \| = \sup_{t \geq 0} \left\{ N_0 e^{-f'_0(\delta + \mu + \gamma + \lambda z)dz} + \int_0^t \lambda \tau e^{-f'_0(\delta + \mu + \gamma + \lambda z)dz} d\tau \right\}
\]

\[
= \sup_{t \geq 0} \left\{ N_0 e^{-f'_0(\delta + \mu + \gamma + \lambda z)dz} + \int_0^t \frac{\lambda \tau}{\delta + \mu + \gamma + \lambda z} (\delta + \mu + \gamma + \lambda z) e^{-f'_0(\delta + \mu + \gamma + \lambda z)dz} d\tau \right\}
\]

\[
\leq \sup_{t \geq 0} \left\{ N_0 e^{-f'_0(\delta + \mu + \gamma + \lambda z)dz} - \frac{\lambda}{\delta + \mu + \gamma + \lambda} e^{-f'_0(\delta + \mu + \gamma + \lambda z)dz} \left|_0^t \right. \right\}
\]

\[
= \sup_{t \geq 0} \left\{ N_0 e^{-f'_0(\delta + \mu + \gamma + \lambda z)dz} + \frac{\lambda}{\delta + \mu + \gamma + \lambda} (1 - e^{-f'_0(\delta + \mu + \gamma + \lambda z)dz}) \right\}
\]

\[
\leq \max \left\{ N_0, \frac{\lambda}{\delta + \mu + \gamma + \lambda} \right\} = N
\]

since \( \lambda_t = \lambda(N_t) \leq \lambda = h(\bar{v}) \ln \left( \frac{1}{1-N} \right) \) by (28) and \( N \) is the unique solution to

\[
N \equiv \left\{ N_0, \frac{h(\bar{v}) \ln \left( \frac{1}{1-N} \right)}{\delta + \mu + \gamma + h(\bar{v}) \ln \left( \frac{1}{1-N} \right)} \right\}.
\]

As \( N < 1 \) if \( N_0 < 1 \) and \( \bar{v} < \infty \), the necessary and sufficient condition for Lipschitz continuity, condition (32), holds if unemployment exists initially, \( N_0 < 1 \).

Furthermore,

\[
\| (Mv)_i (N_0) \| \leq \int_0^\infty \left( \bar{p} - b + \gamma \bar{v} + \max_{h \geq 0} \{ h \bar{v} - c(h) \} \right) e^{-f'_0(r+\delta + \mu + \gamma + \lambda z)dz} dt
\]

\[
\leq \frac{\bar{p} - b + \max_{h \geq 0} \{ h \bar{v} - c(h) \} + \gamma \bar{v}}{r + \delta + \mu + \gamma}
\]

\[
\leq \frac{\left( \bar{p} - b + \max_{h \geq 0} \{ h \bar{v} - c(h) \} \right)}{r + \delta + \mu} = \bar{v} < \infty
\]

from Assumption 2. ■

Note that the result does not rule out multiple equilibria.
4 Summary

References


